Part II A Practicum To Classical Electrodynamics Method

Chapter 1 / Vector Analysis 1
1.1 Vector Operations 1
1.2 Vector Algebra: Triple Products 2
1.3 Position, Displacement, and separation vectors 2
1.4 Problems with Solutions 3
Chapter 2 / Electrostatics 19
2.1 The Electric Field 19
2.2 Divergence and Curl of Electrostatic Fields 20
2.3 Electric Potential 22
2.4 Work and Energy in Electrostatics 24
2.5 Conductors 25
2.6 Problems with Solutions 26
Chapter 3 / Special Techniques 41
3.1 Laplace's Equation 41
3.2 Multipole Expansion 42
3.3 Problems and Solutions 43
Chapter 4 / Electric Fields in Matter 65
4.1 Problems and Solutions 65
Chapter 5 / Magnetostatics 80
5.1 The Lorentz Force Law 80
5.2 Currents 81
5.3 The Biot-Savart Law 83
5.4 The Divergence and Curl of **B** 84
5.5 Magnetic Vector Potential 86
5.6 Multipole Expansion of the Vector Potential 87
5.7 Problems with Solutions 89
Chapter 6 / Magnetic Fields in Matter 100
6.1 Magnetization 100
6.2 Torques and Forces on Magnetic Dipoles 100
6.3 Effect of a Magnetic Field on Atomic Orbits 101
6.4 Magnetization and the Field of a Magnetized Object 102
6.5 The Auxiliary Field H 103
6.6 Linear and Nonlinear Media 104
6.7 Problems with Solutions 107
Chapter 7 / Magnetic Fields in Matter 100
6.1 Magnetization 100
6.2 Torques and Forces on Magnetic Dipoles 100
6.3 Effect of a Magnetic Field on Atomic Orbits 101
6.4 Magnetization and the Field of a Magnetized Object 102
6.5 The Auxiliary Field H 103
6.6 Linear and Nonlinear Media 104
6.7 Problems with Solutions 107

Chapter 1

Vector Analysis

1.1 Vector Operations

The Dot product of two vectors is defined as

$$\mathbf{A} \times \mathbf{B} \equiv AB \sin\theta \hat{\mathbf{n}}$$

where θ is the angle they form when placed tail-to-tail. Note that $\mathbf{A}\cdot\mathbf{B}$ is itself a *scalar* (hence the alternative name **scalar product**). The dot product is *commutative*,

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$$

and *distributive*,

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}$$

if two vectors are parallel, then $\mathbf{A} \cdot \mathbf{B} = AB$. In particular, for any vector $\mathbf{A}$,

$$\mathbf{A} \cdot \mathbf{A} = A^2$$

if $\mathbf{A}$ and $\mathbf{B}$ are perpendicular, then $\mathbf{A}\cdot\mathbf{B} = 0$**.**

The cross product is defined as

$$\mathbf{A} \times \mathbf{B} = AB \sin\theta \hat{\mathbf{n}},$$

where $\hat{\mathbf{n}}$ is a unit vector pointing perpendicular to the plane of $\mathbf{A}$ and $\mathbf{B}$. The cross product is *distributive*,

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) + (\mathbf{A} \times \mathbf{C})$$

but not *commutative*. In fact,

$$(\mathbf{B} \times \mathbf{A}) = -(\mathbf{A} \times \mathbf{B})$$

geometrically, $|A \times B|$ is the area of the parallelogram generated by $\mathbf{A}$ and $\mathbf{B}$. If two vectors are parallel, their cross product is zero. In particular,

$$\mathbf{A} \times \mathbf{A} = 0$$

1.2 Vector Algebra: Triple Products

(i) Scalar triple product: $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$. Geometrically, $|\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})|$ is the volume of the parallelepiped generated by **A, B,** and **C,** since $|\mathbf{B}\times\mathbf{C}|$ is the area of the base, and $\mathbf{A}\cos\theta$is the altitude. Evidently,

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$$

for they all correspond to the same figure. Not that “alphabetical” order is preserved. The “nonalphabetical” triple products,

$$\mathbf{A}\cdot(\mathbf{C}\times\mathbf{B}) = \mathbf{B}\cdot(\mathbf{A}\times\mathbf{C}) = \mathbf{C}\cdot(\mathbf{B}\times\mathbf{A})$$

have the opposite sign. In component form,

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

(ii) Vector Triple Product: $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$. The vector triple product can be simplified by the so called **BAC-CAB** rule,

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$$

1.3 Position, Displacement, and seperation vectors

The location of a point in three dimensions can be described by listing it Cartesian coordinates (x,y,z) . The vector to that point from the origin is called the **position vector**:

$$\mathbf{r} \equiv x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$$

its magnitude is given as

$$r = \sqrt{x^2+y^2+z^2}$$

is the distance to the origin, and

$$\hat{\mathbf{r}} = \frac{\mathbf{r}}{r} = \frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{\sqrt{x^2+y^2+z^2}}$$

is a unit vector pointing radially outward. The **infinitesimal displacement vector**, is given as

$$d\mathbf{l} = dx\hat{\mathbf{x}} + dy\hat{\mathbf{y}} + dz\hat{\mathbf{z}}$$

the **seperation vector** from a source point to the field is given as

$$\tilde{\eta} \equiv \mathbf{r} - \mathbf{r}^{'}$$

its magnitude is

$$\eta = \left| r - r^{'} \right|$$

and a unit vector in the direction from $\mathbf{r}^{'}$ to $\mathbf{r}$ is

$$\hat{\eta} = \frac{\vec{\eta}}{\eta} = \frac{\mathbf{r} - \mathbf{r}^{'}}{\left| r - r^{'} \right|}$$

in Cartesian coordinates,

$$\begin{aligned}
\vec{\eta} &= (x-x')\hat{\mathbf{x}} + (y-y')\hat{\mathbf{y}} + (z-z')\hat{\mathbf{z}} \\
\eta &= \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2} \\
\hat{\eta} &= \frac{(x-x')\hat{\mathbf{x}} + (y-y')\hat{\mathbf{y}} + (z-z')\hat{\mathbf{z}}}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}
\end{aligned}$$

1.4 Problems with Solutions

Problem # 1

The Equation giving a family of ellipsoids is

$$u = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$$

Find the unit vector normal to each point of the surface of these ellipsoids.

Given a three-dimensional surface we can find what the normal vector is by using

$$\hat{u} = \frac{\nabla u}{\parallel \nabla u \parallel}$$

and knowing that

$$\nabla u = \left(\frac{\partial u}{\partial x}\hat{x} + \frac{\partial u}{\partial y}\hat{y} + \frac{\partial u}{\partial z}\hat{z} \right) = \frac{2x}{a^2}\hat{x} + \frac{2y}{b^2}\hat{y} + \frac{2z}{c^2}\hat{z}$$

and for the magnitude of this vector we find

$$\parallel \nabla u \parallel = 2\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}$$

thus the unit vector $\hat{u}$ is given as

$$\hat{u} = \frac{\frac{x}{a^2}\hat{x} + \frac{y}{b^2}\hat{y} + \frac{z}{c^2}\hat{z}}{\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}}$$

Problem # 2

For each of the following vector fields **A** (1) determine whether **A** is constant, (2) find the divergence and the curl **A**, and (3) find the components of **A** in cartesian (expressed in x, y, and z), cylindrical (expressed using ρ, ϕ and z) and spherical coordinates (expressed in r, θ, and ϕ). In all cases a, b, and c are constants.

(a). $\mathbf{A} = a\hat{x} + b\hat{y} + c\hat{z}$

A is constant because any point in the field will give you the same unit vector at that point.

The Divergence of **A** is given by

$$\nabla \cdot \mathbf{A} = \left(\frac{\partial \mathbf{A}}{\partial x} + \frac{\partial \mathbf{A}}{\partial y} + \frac{\partial \mathbf{A}}{\partial z} \right) = \frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} + \frac{\partial c}{\partial z} = 0$$

The curl of **A** is given by

$$\nabla \times \mathbf{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial z} & \frac{\partial}{\partial z} \\ a & b & c \end{vmatrix} = \boxed{0}$$

in cartesian coordinates the components of **A** are given by

$$\boxed{A_x = a \quad A_y = b \quad A_z = c}$$

in cylindrical coordinates we must use a transformation matrix to go from one basis to another, i.e

$$\begin{pmatrix} \hat{\rho} \\ \hat{\phi} \\ \hat{z} \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix}$$

where the 3x3 matrix is the transformation matrix that lets you represent cylindrical coordinates in cartesian coordinates. Thus we find

$$\begin{aligned} \hat{\rho} &= \cos\phi\hat{\mathbf{x}} + \sin\phi\hat{\mathbf{y}} \\ \hat{\phi} &= -\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}} \\ \hat{z} &= \hat{\mathbf{z}} \end{aligned}$$

thus the components are given by

$$\begin{aligned} A_\rho &= \hat{\rho} \cdot \mathbf{A} = a\cos\phi + b\sin\phi \\ A_\phi &= \hat{\phi} \cdot \mathbf{A} = -a\sin\phi + b\cos\phi \\ A_z &= \hat{z} \cdot \mathbf{A} = c \end{aligned}$$

In spherical coordinates we must also use a transformation matrix.

$$\begin{pmatrix} \hat{\mathbf{r}} \\ \hat{\theta} \\ \hat{\phi} \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix}$$

where the 3x3 matrix is the transformation matrix that lets you represent the spherical coordinates in cartesian coordinates, this yields

$$\begin{aligned} \hat{\mathbf{r}} &= \sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}} \\ \hat{\theta} &= \cos\theta\cos\phi\hat{\mathbf{x}} + \cos\theta\sin\phi\hat{\mathbf{y}} - \sin\theta\hat{\mathbf{z}} \\ \hat{\phi} &= -\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}} \end{aligned}$$

thus we find the components to be given by

$$\begin{aligned} A_r &= \hat{\mathbf{r}} \cdot \mathbf{A} = a\sin\theta\cos\phi + b\sin\theta\sin\phi + c\cos\theta \\ A_\theta &= \hat{\theta} \cdot \mathbf{A} = a\cos\theta\cos\phi + b\cos\theta\sin\phi - c\sin\theta \\ A_\phi &= \hat{\phi} \cdot \mathbf{A} = -a\sin\phi + b\cos\phi \end{aligned}$$

\

(b). $\mathbf{A} = a\hat{\rho} + b\hat{\phi} + c\hat{z}$

$\mathbf{A}$ is not constant because as your position changes, so do $\hat{\rho}$ and $\hat{\phi}$. The $\hat{\rho}$ component is the distance from the z axis and the direction is always in the $\hat{\rho}$direction.

The Divergence of $\mathbf{A}$ is given by

$$\nabla\cdot\mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho a) + \frac{1}{\rho}\frac{\partial b}{\partial\phi} + \frac{\partial c}{\partial z} = \boxed{\frac{a}{\rho}}$$

The curl of $\mathbf{A}$ is given by

$$\nabla\times\mathbf{A} = \frac{b}{\rho}\hat{z}$$

in cartesian coordinates we must use

$$\begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{\rho} \\ \hat{\phi} \\ \hat{z} \end{pmatrix}$$

where the 3x3 matrix is the transformation matrix that lets you represent cylindrical coordinates in cartesian coordinates. Thus we find

$$\begin{aligned} \hat{\mathbf{x}} &= \cos\phi\hat{\rho} - \sin\phi\hat{\phi} \\ \hat{\mathbf{y}} &= \sin\phi\hat{\rho} + \cos\phi\hat{\phi} \\ \hat{\mathbf{z}} &= \hat{\mathbf{z}} \end{aligned}$$

thus the components are given by

$$\begin{aligned} A_x &= \hat{\mathbf{x}}\cdot\mathbf{A} = a\cos\phi - b\sin\phi \\ A_y &= \hat{\mathbf{y}}\cdot\mathbf{A} = a\sin\phi + b\cos\phi \\ A_z &= \hat{\mathbf{z}}\cdot\mathbf{A} = c \end{aligned}$$

in cylindrical coordinates the components of $\mathbf{A}$ are given by

$$A_\rho = a \quad A_\phi = b \quad A_z = c$$

In spherical coordinates we must also use a transformation matrix.

$$\begin{pmatrix} \hat{\mathbf{r}} \\ \hat{\theta} \\ \hat{\phi} \end{pmatrix} = \begin{pmatrix} \sin\theta & 0 & \cos\theta \\ -\cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \hat{\rho} \\ \hat{\phi} \\ \hat{z} \end{pmatrix}$$

where the 3x3 matrix is the transformation matrix that lets you represent the spherical coordinates in cylindrical coordinates, this yields

$$\begin{aligned} \hat{\mathbf{r}} &= \sin\theta\hat{\rho} + \cos\theta\hat{\mathbf{z}} \\ \hat{\theta} &= -\cos\theta\hat{\rho} - \sin\theta\hat{\mathbf{z}} \\ \hat{\phi} &= \hat{\phi} \end{aligned}$$

thus we find the components to be given by

$$\begin{aligned} A_r &= \hat{\mathbf{r}}\cdot\mathbf{A} = a\sin\theta + c\cos\theta \\ A_\theta &= \hat{\theta}\cdot\mathbf{A} = -a\cos\theta - c\sin\theta \\ A_\phi &= \hat{\phi}\cdot\mathbf{A} = b \end{aligned}$$

(c). $\mathbf{A} = a\hat{r} + b\hat{\theta} + c\hat{\phi}$

A is not constant with the same argument as above except that $\hat{r}, \hat{\theta}$, and ϕ all change according to where the point is.

The Divergence of **A** is given by

$$\begin{aligned} \nabla \cdot \mathbf{A} &= \frac{1}{r^2}\frac{\partial}{\partial l}(r^2 a) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta b) + \frac{1}{r\sin\theta}\frac{\partial c}{\partial\phi} \\ &= \boxed{\frac{2a}{r} + \frac{c}{r}\cot\theta} \end{aligned}$$

The curl of **A** is given by

$$\begin{aligned} \nabla \times \mathbf{A} &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta c) - \frac{\partial b}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial a}{\partial\phi} - \frac{\partial}{\partial r}(rc)\right]\hat{\theta} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rb) - \frac{\partial a}{\partial\theta}\right]\hat{\phi} \\ &= \boxed{\frac{c}{r}\cot\theta\hat{\mathbf{r}} - \frac{c}{r}\hat{\theta} + \frac{b}{r}\hat{\phi}} \end{aligned}$$

in cartesian coordinates the components of **A** are given by

$$\begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{r}} \\ \hat{\theta} \\ \hat{\phi} \end{pmatrix}$$

where the 3x3 matrix is the transformation matrix that lets you represent the spherical coordinates in cartesian coordinates, this yields

$$\begin{aligned} \hat{\mathbf{x}} &= \sin\theta\cos\phi\hat{\mathbf{r}} + \cos\theta\cos\phi\hat{\theta} + -\sin\phi\hat{\phi} \\ \hat{\mathbf{y}} &= \sin\theta\sin\phi\hat{\mathbf{r}} + \cos\theta\sin\phi\hat{\theta} - \cos\phi\hat{\phi} \\ \hat{\mathbf{z}} &= \cos\theta\hat{\mathbf{r}} + -\sin\theta\hat{\theta} \end{aligned}$$

thus we find the components to be given by

$$\begin{aligned} A_x &= \hat{\mathbf{x}} \cdot \mathbf{A} = a\sin\theta\cos\phi + \cos\theta\cos\phi\phi + -c\sin\theta \\ A_y &= \hat{\mathbf{y}} \cdot \mathbf{A} = a\sin\theta\sin\phi + b\cos\theta\sin\phi - c\cos\phi\theta \\ A_z &= \hat{\mathbf{z}} \cdot \mathbf{A} = a\cos\theta - b\sin\theta \end{aligned}$$

in cylindrical coordinates the components of **A** are given by

$$\begin{pmatrix} \hat{\rho} \\ \hat{\phi} \\ \hat{z} \end{pmatrix} = \begin{pmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{r}} \\ \hat{\theta} \\ \hat{\phi} \end{pmatrix}$$

where the 3x3 matrix is the transformation matrix that lets you represent the spherical coordinates in cylindrical coordinates, this yields

$$\begin{aligned} \hat{\rho} &= \sin\theta\hat{\mathbf{r}} + \cos\theta\hat{\theta} \\ \hat{\phi} &= \hat{\phi} \\ \hat{\mathbf{z}} &= \cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\theta} \end{aligned}$$

thus we find the components to be given by

$$\begin{aligned} A_\rho &= \hat{\rho} \cdot \mathbf{A} = a \sin\theta + b \cos\theta \\ A_\phi &= \hat{\phi} \cdot \mathbf{A} = c \\ A_z &= \hat{\mathbf{z}} \cdot \mathbf{A} = a \cos\theta - b \sin\theta \end{aligned}$$

and in spherical coordinates they are given by

$$A_r = a \quad A_\theta = b \quad A_\phi = c$$

Problem # 3
Show that

$$\int_V \nabla f \, dV = \oint_A f \, d\mathbf{A}$$

where A is the area of the closed surface bounding the volume V. Hint: multiply each side by a constant vector and use calculus theorems.

If we multiply both sides by a constant vector field **A** we get

$$\mathbf{A} \cdot \int_V \nabla f \, dV = \mathbf{A} \cdot \oint_A f \cdot d\mathbf{a}$$

if we work on the left-hand side, along with using equation 5 from the front of Griffith's we find

$$\int_V (\mathbf{A} \cdot \nabla f) dV = \int_V \nabla \cdot (f\mathbf{A}) dV - \int_V f(\nabla \cdot \mathbf{A}) dV$$

we know that the right hand term goes to zero because the gradient of a constant vector field is 0. Invoking the Divergence theorem on the left hand term we get

$$\int_V \nabla \cdot (f\mathbf{A}) dV = \oint_A f\mathbf{A} \cdot d\mathbf{a}$$

we finally find

$$\begin{aligned} \mathbf{A} \cdot \int_V \nabla f dV &= \oint_A f\mathbf{A} \cdot d\mathbf{a} \\ \mathbf{A} \cdot \int_V \nabla f dV &= \oint_A \mathbf{A} \cdot (f d\mathbf{a}) \\ \mathbf{A} \cdot \int_V \nabla f dV &= \mathbf{A} \cdot \oint_A f d\mathbf{a} \end{aligned}$$

thus we have

$$\boxed{\int_V \nabla f \, dV = \oint_A f \, d\mathbf{A}}$$

Problem # 4
Show that the velocity ($d\mathbf{r}/dt = \dot{\mathbf{r}}$) is expressed in cylindrical coordinates as $\dot{\rho}\hat{\rho} + \rho\dot{\phi}\hat{\phi} + \dot{z}\hat{z}$ and in spherical coordinates as $\dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\theta} + r\sin\theta\dot{\phi}\hat{\phi}$.

We know that the velocity can also be written as

$$\dot{\mathbf{r}} = \frac{dx}{dt}\hat{x} + \frac{dy}{dt}\hat{y} + \frac{dz}{dt}\hat{z}$$

and we know that in cylindrical coordinates

$$\begin{aligned} x &= \rho\cos\phi \quad \frac{d}{dt}(\rho\cos\phi) = \dot{\rho}\cos\phi - \dot{\phi}\rho\sin\phi \\ y &= \rho\sin\phi \quad \frac{d}{dt}(\rho\sin\phi) = \dot{\rho}\sin\phi + \dot{\phi}\rho\cos\phi \\ z &= z \quad \frac{d}{dt}(z) = \dot{z} \end{aligned}$$

and we also know that in cylindrical cooridinates

$$\begin{aligned} \hat{x} &= \cos\phi\hat{\rho} - \sin\phi\hat{\phi} \\ \hat{y} &= \sin\phi\hat{\rho} + \cos\phi\hat{\phi} \\ z &= \hat{z} \end{aligned}$$

thus we can write the velocity as

$$\begin{aligned} \dot{\mathbf{r}} &= (\dot{\rho}\cos\phi - \dot{\phi}\rho\sin\phi)(\cos\phi\hat{\rho} - \sin\phi\hat{\phi}) + (\dot{\rho}\sin\phi + \dot{\phi}\rho\cos\phi)(\sin\phi\hat{\rho} + \cos\phi\hat{\phi}) + \dot{z}\hat{z} \\ &= \dot{\rho}\cos^2\phi\hat{\rho} + \dot{\phi}\sin^2\phi\hat{\phi} + \dot{\rho}\sin^2\phi\hat{\rho} + \dot{\phi}\cos^2\phi\hat{\phi} + \dot{z}\hat{z} \\ &= \boxed{\dot{\rho}\hat{\rho} + \rho\dot{\phi}\hat{\phi} + \dot{z}\hat{z}} \end{aligned}$$

in Spherical coordinates we can apply the same technique.and we know that in spherical coordinates

$$\begin{aligned} x &= r\sin\theta\cos\phi \quad \frac{d}{dt}(r\sin\theta\cos\phi) = \dot{r}\sin\theta\cos\phi + \dot{\theta}r\cos\theta\cos\phi - \dot{\phi}r\sin\theta\sin\phi \\ y &= r\sin\theta\sin\phi \quad \frac{d}{dt}(r\sin\theta\sin\phi) = \dot{r}\sin\theta\sin\phi + \dot{\theta}r\cos\theta\sin\phi + \dot{\phi}r\sin\theta\cos\phi \\ z &= r\cos\theta \quad \frac{d}{dt}(r\cos\theta) = \dot{r}\cos\theta - \dot{\theta}\sin\theta \end{aligned}$$

and we also know that in cylindrical cooridinates

$$\begin{aligned} \hat{x} &= \cos\phi\hat{\rho} - \sin\phi\hat{\phi} \\ \hat{y} &= \sin\phi\hat{\rho} + \cos\phi\hat{\phi} \\ \hat{z} &= \hat{z} \end{aligned}$$

thus we can write the velocity as being (minus some algebra)

$$\dot{\mathbf{r}} = \frac{dx}{dt}\hat{x} + \frac{dy}{dt}\hat{y} + \frac{dz}{dt}\hat{z} = \boxed{\dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\theta} + r\sin\theta\dot{\phi}\hat{\phi}}$$

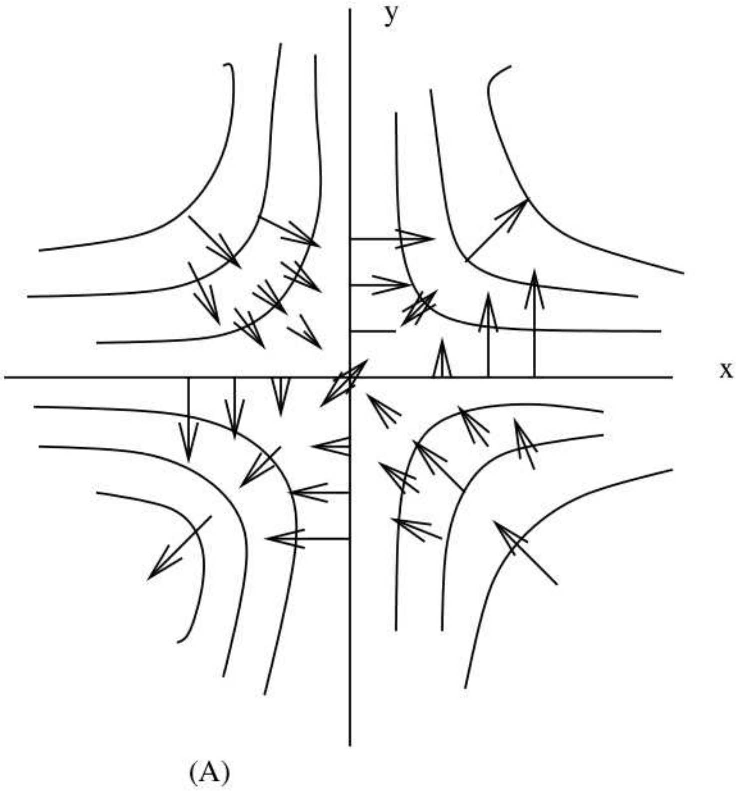
Problem # 5
Here we will practice the use of field differential operators and visualize some fields.

(a). Find the gradient of the scalar potential $\phi(x,y,z) = \alpha xy$. Provide a clear sketch of the countour lines of ϕ in the $\hat{x} - \hat{y}$ plane and a representation of its gradient field. Such a field is known as a radial quadrupole field, and is used in the focusing of charged particle beams and of dipolar (electric or magnetic) particles.

The gradient of ϕ is given by

$$\begin{aligned}\nabla\phi &= \alpha\left(\frac{\partial}{\partial x}(xy)+\frac{\partial}{\partial y}(xy)+\frac{\partial}{\partial z}(xy)\right)\\ &= \boxed{\alpha(y\hat{x}+x\hat{y})}\end{aligned}$$

The sketch is given as



(b). Provide a clear sketch (in the $\hat{x}-\hat{y}$ plane) of the vector field expressed in cylindrical coordinates as $\mathbf{E}=\alpha\rho\hat{\rho}$ where α is a constant. Calculate its divergence. If $\mathbf{E}$ is an electric field, what charge distribution generates it?

The gradient of $\mathbf{E}$ is given by

$$\begin{aligned}\nabla\mathbf{E} &= \alpha\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}(\alpha\rho^2)\hat{\rho}\right)\\ &= \boxed{2\alpha\hat{\rho}}\end{aligned}$$

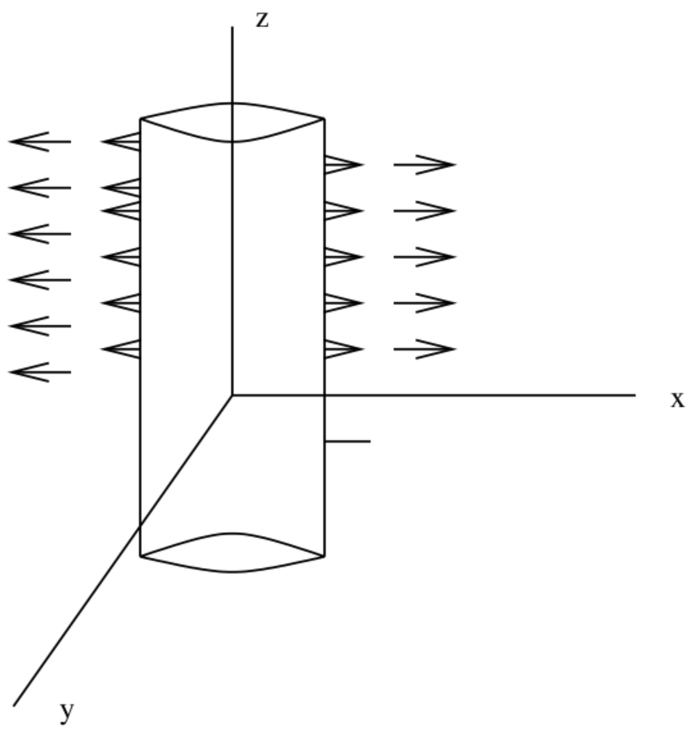
The divergence of $\mathbf{E}$ is given by

$$\nabla\cdot\mathbf{E}=\boxed{2\alpha}$$

which is the direction of most change. The charge distribution that produces this charge can be found by using Gauss's law

$$\nabla \cdot \mathbf{E} = \frac{q_{enc}}{\varepsilon_0} = 2\alpha \quad \Rightarrow \quad \boxed{q_{enc} = 2\alpha\varepsilon_0}$$

The sketch is given by



(B)

(c). Consider a large sheet placed on a turntable that is set rotating at angular velocity Ω. What vector field $\mathbf{v}$ describes the local velocity on this sheet? Provide a clear sketch of this field. What is the curl of this vector field (now expressed as a vector in a three-dimensional space, i.e including a coordinate for the z axis)? If a magnetic vector potential $\mathbf{A}$ is defined with the functional form of $\mathbf{v}$ (just a different constant out front), what magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ does it represent.

The vector describing this system is given by

$$\mathbf{v} = r\Omega\hat{\phi}$$

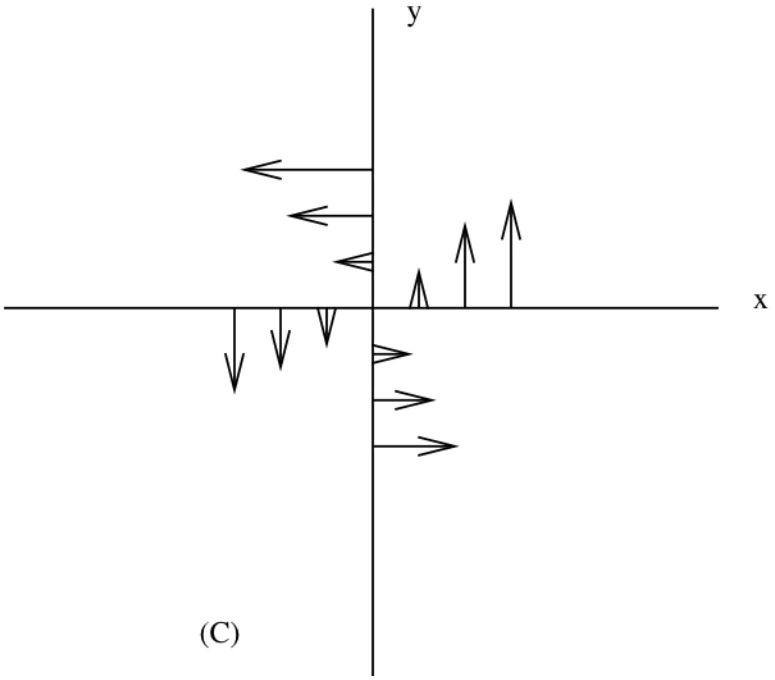
where r is the distance from the origin and θ is the angle between any lines originating from the origin. The curl is given by

$$\nabla \times \mathbf{v} = \frac{1}{r}\frac{\partial}{\partial r}(r^2\Omega)\hat{\mathbf{z}} = \boxed{2\Omega\hat{\mathbf{z}}}$$

The magnetic field is given by

$$\mathbf{B} = \nabla \times \mathbf{v} = 2\Omega\hat{\mathbf{z}}$$

The sketch is given by



(d). Now generate new vector field by taking your vector field **v** obtained in part (c) above, and adding (1) a constant field, and (2) a radial quadrupole field from part (a). Provide clear sketches of these two fields and compare with that of **v.** Show explicitly that the curl of these three fields is the same. Note that these three fields all represent forms of a magnetic vector potential that describes the same magnetic field.

We can define a constant vector field as

$$\mathbf{k} = a\hat{\mathbf{x}} + b\hat{\mathbf{y}} + c\hat{\mathbf{z}}$$

which can also be represented as

$$\mathbf{k} = (a\cos\phi + b\sin\theta\phi)\hat{\rho} + (-a\sin\phi + b\cos\phi)\hat{\phi} + c\hat{\mathbf{z}}$$

so now we can add these two vectors

$$\mathbf{v} + \mathbf{k} = \boxed{(a\cos\phi + b\sin\theta\phi)\hat{\rho} + (-a\sin\phi + b\cos\phi + r\Omega)\hat{\phi} + c\hat{\mathbf{z}}}$$

From part (a) we found

$$\begin{aligned} \nabla\phi &= \alpha(y\hat{\mathbf{x}} + x\hat{\mathbf{y}}) \\ &= \alpha(y\cos\phi + x\sin\phi)\hat{\rho} + \alpha(x\cos\phi - y\sin\phi)\hat{\phi} \end{aligned}$$

adding up these vectors yield

$$\mathbf{v} + \nabla\phi = \alpha(y\cos\phi + x\sin\phi)\hat{\rho} + [r\Omega + \alpha(x\cos\phi - y\sin\phi)\hat{\phi}$$

To find the curl we use

$$\nabla \times (\mathbf{v} + \mathbf{k}) = \nabla \times \mathbf{v} + \nabla \times \mathbf{k}$$

the right hand term goes to 0 and the left hand term is

$$\nabla \times \mathbf{v} = 2\Omega\hat{\mathbf{z}}$$

also

$$\nabla \times (\mathbf{v} + \nabla \phi) = \nabla \times \mathbf{v} + \nabla \times \nabla \phi$$

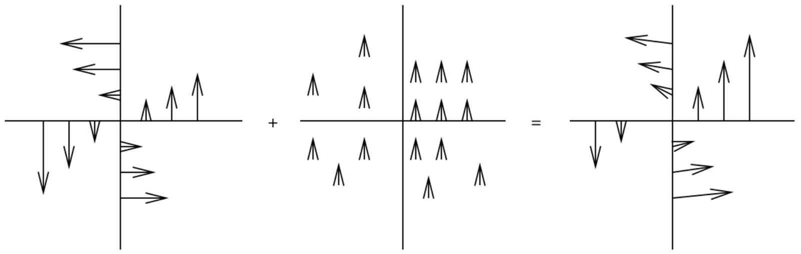
the right hand term is also 0, so we get

$$\nabla \times \mathbf{v} = 2\Omega \hat{\mathbf{z}}$$

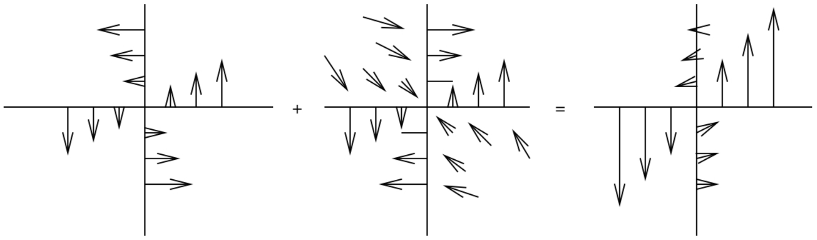
we have just showed explicitly that all the curls are the same,. For the firs sketch we can define a constant vector field as

$$\mathbf{k} = u\hat{\mathbf{y}}$$

and then add these two vector fields, as show in the diagram part (a), the second diagram (b) is the sum of our original vector field added to the quadrupole field given by (a).



(a)



(b)

Problem # 6

(a) Show that

$$\int_s f(\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \int_s [\mathbf{A} \times (\nabla f)] \cdot d\mathbf{a} + \oint_p f\mathbf{A} \cdot d\mathbf{l}$$

Using equation 7 from the front cover of Griffiths's we find

$$\int_s f(\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \int_s [\nabla \times (f\mathbf{A})] \cdot d\mathbf{a} + \int_s [\mathbf{A} \times (\nabla f)] \cdot d\mathbf{a}$$

invoking Stoke's theorem on the left hand term yield

$$\int_s [\nabla \times (f\mathbf{A})] \cdot d\mathbf{a} = \oint_p f\mathbf{A} \cdot d\mathbf{l}$$

thus we find

$$\boxed{\int_s f(\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \int_s [\mathbf{A} \times (\nabla f)] \cdot d\mathbf{a} + \oint_p f\mathbf{A} \cdot d\mathbf{l}}$$

(b) Show that

$$\int_V \mathbf{B}\cdot(\nabla\times\mathbf{A})d\tau = \int_V \mathbf{A}\times(\nabla\times\mathbf{B})d\tau + \oint_s (\mathbf{A}\times\mathbf{B})\cdot d\mathbf{a}$$

Using equation 6 from thr front cover of Griffiths we find

$$\int_V \mathbf{B}\cdot(\nabla\times\mathbf{A})d\tau = \int_V \nabla\cdot(\mathbf{A}\times\mathbf{B})d\tau + \int_V \mathbf{A}\cdot(\nabla\times\mathbf{B})d\tau$$

Using Gauss's theorem on the left hand term we find

$$\int_V \nabla\cdot(\mathbf{A}\times\mathbf{B})d\tau = \oint_s (\mathbf{A}\times\mathbf{B})\cdot d\mathbf{a}$$

Thus we have

$$\boxed{\int_V \mathbf{B}\cdot(\nabla\times\mathbf{A})d\tau = \int_V \mathbf{A}\times(\nabla\times\mathbf{B})d\tau + \oint_s (\mathbf{A}\times\mathbf{B})\cdot d\mathbf{a}}$$

Problem # 7

(a) Let $\mathbf{F_1} = x^2\hat{\mathbf{z}}$ and $\mathbf{F}_2 = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$. Calculate the divergence and curl of $\mathbf{F}_1$ and $\mathbf{F}_2$. Which one can be written as a gradient of a scalar? Find a scalr potential that does the job. Which one can be written as a curl of a vector? Find a suitable vector potential.

The divergence of $\mathbf{F}_1$ is given by

$$\nabla\cdot\mathbf{F}_1 = \frac{\partial}{\partial z}x^2 = 0$$

and the curl is given by

$$\nabla\times\mathbf{F}_1 = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & x^2 \end{vmatrix} = \frac{\partial}{\partial x}x^2\hat{\mathbf{y}} = -2x\hat{\mathbf{y}}$$

The divergence of $\mathbf{F}_2$ is given by

$$\nabla\cdot\mathbf{F}_2 = \frac{\partial}{\partial x}x + \frac{\partial}{\partial zy}y + \frac{\partial}{\partial z}z = 3$$

and the curl is given by

$$\nabla\times\mathbf{F}_2 = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0$$

We know that $\mathbf{F}_1$ can be written as the curl of a vector potential because it is divergen-less or "**solenoidal**", to find the vector potential we do

$$\mathbf{F}_1 = \nabla\times V$$

We need to find the vector potential V, so we can find it using

$$\mathbf{F}_1 = \nabla\times V = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \frac{\partial A_y}{\partial x}\hat{\mathbf{z}} = x^2\hat{\mathbf{z}}$$

thus A_y is given by

$$A_y = \frac{x^3}{3}$$

and the vector potential is given as

$$V = \boxed{a\hat{\mathbf{x}} + \frac{x^3}{3}\hat{\mathbf{y}} + b\hat{\mathbf{z}}}$$

where a and b are constants. We know that $\mathbf{F}_2$ can be written as the gradient of a scalar potential because it is **“curl-less**”. we find the scalar potential by using

$$\begin{aligned} \mathbf{F}_2 &= -\nabla V \\ &= -\nabla\left[\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2}\right] \end{aligned}$$

thus we know that the scalar potential is

$$V = \boxed{-\left[\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2}\right]}$$

(b) Show that $\mathbf{F}_3 = yz\hat{\mathbf{x}} + zx\hat{\mathbf{y}} + xy\hat{\mathbf{z}}$ can be written both as the gradient of a scalar and as the curl of a vector. Find scalar and vector potentials for this function.

First we must show that $\mathbf{F}_3$ is is both divergen-less and curl-less, to show divergen-less and find the vector potential we use

$$\nabla \cdot \mathbf{F}_3 = \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}(xy) = 0$$

thus we must use

$$\mathbf{F}_3 = \nabla \times V = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\mathbf{x}} - \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)\hat{\mathbf{y}} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\mathbf{z}}$$

Solving for $A_{x,}, A_y$, and A_z we find

$$\begin{aligned} yz\hat{\mathbf{x}} &= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\mathbf{x}} \;\; A_z = \frac{zy^2}{2} \\ zx\hat{\mathbf{y}} &= \left(\frac{\partial A_x}{\partial x} - \frac{\partial A_z}{\partial z}\right)\hat{\mathbf{y}} \;\; A_x = \frac{yz^2}{2} \\ xy\hat{\mathbf{z}} &= \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\mathbf{z}} \;\; A_y = \frac{xy^2}{2} \end{aligned}$$

Thus the vector potential is

$$V = \boxed{\frac{zy^2}{2}\hat{\mathbf{x}} + \frac{xy^2}{2}\hat{\mathbf{y}} + \frac{zy^2}{2}\hat{\mathbf{z}}}$$

For the scalar potential we do

$$\nabla \times \mathbf{F}_3 = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix} = 0$$

by inspection. The scalar potential is given by

$$\mathbf{F}_3 = yz\hat{\mathbf{x}} + zx\hat{\mathbf{y}} + xy\hat{\mathbf{z}} = -\nabla V$$

and we find

$$V = -xyz$$

thus the vector can be written as

$$\mathbf{F}_3 = \boxed{-\nabla(xyz)}$$

Problem # 8

Check the divergence theorem for the function

$$\mathbf{v} = r^2 \sin\theta \hat{\mathbf{r}} + 4r^2 \cos\theta \hat{\theta} + r^2 \tan\theta \hat{\phi}$$

using the volume of the "ice-cream cone" shown on page 56 Fig. 1.52 (the top surface is spherical, with radius R and centered at the origin).

The divergence theorem states

$$\int_V \nabla \cdot \mathbf{v} dV = \oint_A \mathbf{v} \cdot d\mathbf{a}$$

the left hand side can be solved by taking a volume integral of **v.**

$$\begin{aligned} \nabla \cdot \mathbf{v} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^4 \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(4r^2 \sin\theta \cos\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}(r^2 \tan\theta) \\ &= 4r\sin\theta + \frac{4r}{\sin\theta}\left[(\cos\theta)^2 - (\sin\theta)^2\right] \\ &= 4r\sin\theta + 4r\left[\frac{1}{\sin\theta} - 2(\sin\theta)\right] = 4r\frac{(\cos\theta)^2}{\sin\theta} \end{aligned}$$

and the integral is given by

$$\int_V \nabla \cdot \mathbf{v} dV = \int_0^{2\pi} d\phi \int_0^R 4r^3 dr \int_0^{\pi/6} (\cos\theta)^2 d\theta$$

which yields

$$\int_V \nabla \cdot \mathbf{v} dV = \boxed{2\pi R^4 \left[\frac{\pi}{12} + \frac{\sqrt{3}}{8}\right]}$$

Now for the area integral must look at the the cone and the surface separately

$$\begin{aligned} \oint_A \mathbf{v} \cdot d\mathbf{a} &= \int_0^{2\pi} R^4 d\phi \int_0^{\pi/6} (\sin\theta)^2 d\theta + \frac{1}{2}\int_0^{2\pi} d\phi \int_0^R \sin(\pi/6)\cos(\pi/6) 4r^3 dr \\ &= 2\pi R^4 \left[\frac{\pi}{12} - \frac{\sqrt{3}}{8}\right] + \pi R^4 \left(\frac{\sqrt{3}}{4}\right) \\ &= \boxed{2\pi R^4 \left[\frac{\pi}{12} + \frac{\sqrt{3}}{8}\right]} \end{aligned}$$

Problem # 9

A little more practice on vector analysis: consider the vector function $\mathbf{F}(\mathbf{r}) = \hat{\phi}$ where $\hat{\phi}$ is defined as conventional for spherical coordinates.

(a). Calculate the line integral

$$\oint_c \mathbf{F} \cdot d\mathbf{l}$$

where C is a circle of radius ρ in the $x-y$ plane, centered at the origin, and where the integral is evaluated with $d\mathbf{l}$ oriented counter clockwise.

from the diagram we can see that

$$d\mathbf{l} = \rho d\phi\hat{\phi}$$

given that $\mathbf{F}(\mathbf{r}) = \hat{\phi}$ we find

$$\oint_c \mathbf{F} \cdot d\mathbf{l} = \int_0^{2\pi} \hat{\phi} \cdot (\rho d\phi\hat{\phi}) = \int_0^{2\pi} \rho d\phi = \boxed{2\pi\rho}$$

(b). Calculate the surface integral

$$\int_H (\nabla \times \mathbf{F}) \cdot d\mathbf{a}$$

where the surface H is a hemisphere that is above and bounded by the curve C used for part (a). The surface integral is calculated with $d\mathbf{a}$ oriented outward.

from the diagram we can see that

$$d\mathbf{a} = \rho^2 \sin\theta d\theta d\phi\hat{\mathbf{r}}$$

and to find what the curl of $\mathbf{F}$ we do

$$\nabla \times \mathbf{F} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta)\right]\hat{\mathbf{r}} - \left[\frac{1}{r}\frac{\partial}{\partial r}(r)\right]\hat{\phi} = \frac{\cot\theta}{r}\hat{\mathbf{r}} - \frac{1}{r}\hat{\phi}$$

in our case $r = \rho$ thus we have

$$\nabla \times \mathbf{F} = \frac{1}{\rho}[\cot\theta\hat{\mathbf{r}} - \hat{\phi}]$$

thus the integral is given by

$$\int_H (\nabla \times \mathbf{F}) \cdot d\mathbf{a} = \int_H \frac{1}{\rho}[\cot\theta\hat{\mathbf{r}} - \hat{\phi}] \cdot \rho^2 \sin\theta d\theta d\phi\hat{\mathbf{r}} = \rho\int_0^{\pi/2} \cos\theta d\theta \int_0^{2\pi} d\phi = \boxed{2\pi\rho}$$

(c). Calculate the surface integral

$$\int_D (\nabla \times \mathbf{F}) \cdot d\mathbf{a}$$

where the surface D is now the disk in the $x-y$ plane that is bounded by the curve C used for part (a). The surface integral is calculated with $d\mathbf{a}$ oriented upward.

from the diagram we can see that

$$d\mathbf{a} = \rho dr d\phi\hat{\mathbf{z}} = \rho dr d\phi(\cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\phi}) \quad \theta = \frac{\pi}{2}$$

and to find what the curl is we do

$$\nabla \times \mathbf{F} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta)\right]\hat{\mathbf{r}} - \left[\frac{1}{r}\frac{\partial}{\partial r}(r)\right]\hat{\phi} = \frac{\cot\theta}{r}\hat{\mathbf{r}} - \frac{1}{r}\hat{\phi}$$

in our case $r = \rho$ thus we have

$$\nabla \times \mathbf{F} = \frac{1}{\rho}[\cot\theta \hat{\mathbf{r}} - \hat{\phi}]$$

thus the integral is given by

$$\begin{aligned}
\int_D (\nabla \times \mathbf{F}) \cdot d\mathbf{a} &= \int_D \frac{1}{\rho}[\cot\theta \hat{\mathbf{r}} - \hat{\phi}] \cdot \rho dr d\phi (\cos\theta \hat{\mathbf{r}} - \sin\theta \hat{\phi}) \\
&= \int_0^{2\pi} \int_0^{\rho} \left[\frac{\cos^2\theta + \sin^2\theta}{\sin\theta} \right] dr d\phi \\
&= \int_0^{2\pi} d\phi \int_0^{\rho} dr = \boxed{2\pi\rho}
\end{aligned}$$

(d). Verify that Stoke's theorem holds for both surfaces H and D.

Stokes theorem states

$$\oint \mathbf{F} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{a} = 2\pi\rho$$

thus Stoke's theorem holds.

Problem # 10

Let $\mathbf{H}(\mathbf{r}) = x^2 y \hat{\mathbf{x}} + y^2 z \hat{\mathbf{y}} + z^2 x \hat{\mathbf{z}}$. Find an irrotational function $\mathbf{F}(\mathbf{r})$ and a sonisoidal function $\mathbf{G}(\mathbf{r})$ such that $\mathbf{H} = \mathbf{F} + \mathbf{G}$.

We know that an irrotational function can be described as

$$\nabla \times \mathbf{F} = 0$$

and a sonisoidal function is described as

$$\nabla \cdot \mathbf{G} = 0$$

thus we know that

$$\nabla \times \mathbf{H} = \nabla \times \mathbf{F} + \nabla \times \mathbf{G} = \nabla \times \mathbf{G}$$

and we know that

$$\nabla \times \mathbf{H} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y & y^2 z & z^2 x \end{vmatrix} = -y^2 \hat{\mathbf{x}} - z^2 \hat{\mathbf{y}} - x^2 \hat{\mathbf{z}}$$

thus we can find $\mathbf{G}$ by

$$\nabla \times \mathbf{G} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ G_x & G_y & G_z \end{vmatrix} = \left(\frac{\partial G_z}{\partial y} - \frac{\partial G_y}{\partial z} \right) \hat{\mathbf{x}} - \left(\frac{\partial G_z}{\partial x} - \frac{\partial G_x}{\partial z} \right) \hat{\mathbf{y}} + \left(\frac{\partial G_y}{\partial x} - \frac{\partial G_x}{\partial y} \right) \hat{\mathbf{z}}$$

looking at the following terms we find

$$\frac{\partial G_z}{\partial y} = -y^2 \quad \frac{\partial G_x}{\partial z} = -z^2 \quad \frac{\partial G_y}{\partial x} = -x^2$$

thus

$$G_z = -\frac{y^3}{3} \quad G_x = -\frac{z^3}{3} \quad G_y = -\frac{x^3}{3}$$

thus we find **G** to be

$$\mathbf{G} = -\frac{z^3}{3}\hat{\mathbf{x}} - -\frac{x^3}{3}\hat{\mathbf{y}} - -\frac{y^3}{3}\hat{\mathbf{z}}$$

since we know what **H** and **G** are we can easily find **F**

$$\mathbf{F} = \mathbf{H} - \mathbf{G} = \left(x^2y + \frac{z^3}{3}\right)\hat{\mathbf{x}} + \left(y^2z + \frac{x^3}{3}\right)\hat{\mathbf{y}} + \left(z^2x + \frac{y^3}{3}\right)\hat{\mathbf{z}}$$

and it is seen that

$$\begin{aligned} \nabla \times \mathbf{F} &= 0 \\ \nabla \cdot \mathbf{G} &= 0 \end{aligned}$$

and by construction we know that

$$\mathbf{H} = \mathbf{F} + \mathbf{G}$$

Chapter 2

Electrostatics

2.1 The Electric Field

2.1.1 Coulomb's Law

The fundamental problem electromagnetic theory hopes to solve is this: We have some electric charges $q_1, q_2, q_3, ...$ (call them **source charges**); what force do they exert on another charge, Q (call it the **test charge**). The force on the test charge Q due to a single point charge q which is at rest a distance r' away is given by **Coulomb's Law**

$$\mathbf{F} = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{r^2}\hat{\mathbf{r}}'' \tag{2.1}$$

The cosntant ε_0 is called the **permitivity of free space**. In SI units, where force is in Newtons (N), distance in meters (m), and charge in Coulombs (C)

$$\varepsilon_0 = 8.85 \times 10^{-12}\frac{\mathrm{C}^2}{\mathrm{N}\cdot\mathrm{m}^2}$$

In words, the force is proportional to the product of the charges and inversely proportional to the square of the seperation distance. As always

$$\mathbf{r}'' = \mathbf{r} - \mathbf{r}'$$

where $\mathbf{r}''$ is the seperation vector from the location of q to Q, r is the magnitude and $\hat{\mathbf{r}}''$ is the direction. The force points along the line from q to Q; it is repulsive if q and Q have the same sign, and attractive if their signs are different. Coulomb's law and the principle of superposition constitute the physical input for electrostatics- the rest, except for some special properties of matter, is mathematically elaborationof these fundemental rules.

2.1.2 The Electric Field

If we have several point charges $q_1, q_2, ..., q_n$ at distances $r_1, r_2, ..., r_n$ from Q, the total force on Q is evidently

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + ... = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1 Q}{r_1^2}\hat{\mathbf{r}}_1 + \frac{q_2 Q}{r_2^2}\hat{\mathbf{r}}_2 + ..\right)$$

or

$$\mathbf{F} = Q\mathbf{E}$$

where

$$\boxed{\mathbf{E}(\mathbf{r}) \equiv \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i^2}\hat{\mathbf{r}}_i} \tag{2.2}$$

E is called the **electric field** of the source charges.

2.1.3 Continuous Charge Distributions

Our definition of the electric field Equation 2.2 assumes that the source of the field is a collection of discrete point charges q_i. If, instead, the charge is distributed continuously over some region, the sum becomes an integral

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{1}{r^2}\hat{\mathbf{r}}dq$$

If the charge is spread out along a line, with charge-per-unit length λ, then $dq = \lambda dl'$ (where dl' is an element of length along the line); if the charge is smeared out over a surface, with charge-per-unit are σ, the $dq = \sigma da'$, (where da' is an element of area on the surface) and if the charge fills a volume, with charge-per-unit volume ρ then $dq = \rho d\tau'$ (where $d\tau'$ is an element of volume)

$$dq \rightarrow \lambda dl' \sim \sigma da' \sim \rho d\tau'$$

thus the electric field of a line charge is

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\lambda(\mathbf{r}')}{r^2}\hat{\mathbf{r}}dl'$$

for a surface charge

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma(\mathbf{r}')}{r^2}\hat{\mathbf{r}}da'$$

and for a volume charge

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{r}')}{r^2}\hat{\mathbf{r}}d\tau' \tag{2.3}$$

Equation 2.3 itself is often referred to as Coulomb's law, because it is such a short step from the original, and because a volume charge is in a sense the most general and realistic case.

2.2 Divergence and Curl of Electrostatic Fields

2.2.1 Field lines, Flux, and Gauss's Law

The flux of **E** through a surface S is given as

$$\Phi_E \equiv \int_S \mathbf{E}\cdot d\mathbf{a}$$

it is a measure of the "number of field lines" passing through S. This suggests that the flux through any closed surface is a measure of the total charge inside. This is in essence Gauss's Law. In the case of a point charge q at the origin, the flux **E** through a sphere of radius r is

$$\oint \mathbf{E}\cdot d\mathbf{a} = \int \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r^2}\hat{\mathbf{r}}\right)\cdot(r^2\sin\theta d\theta d\phi\hat{\mathbf{r}} = \frac{q}{\varepsilon_0}$$

For any closed surface, then

$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\varepsilon_0} Q_{\text{enc}}$$

where Q_{enc} is the total charge enclosed within the surface. As it stands, Gauss's law is an integral equation, but we can readily turn it into a differential one, by applying the divergence theorem

$$\oint_S \mathbf{E} \cdot d\mathbf{a} = \int_V (\nabla \cdot \mathbf{E}) d\tau$$

rewriting Q_{enc} in terms of the charge density ρ, we have

$$Q_{\text{enc}} = \int_V \rho d\tau$$

so Gauss's law becomes

$$\int_V (\nabla \cdot \mathbf{E}) d\tau = \int_V \left(\frac{\rho}{\varepsilon_0} \right) d\tau$$

and since this holds for any volume the integrand must be equal

$$\boxed{\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}} \tag{2.4}$$

Equation 2.4 is **Gauss's law in differential form**.

2.2.2 The Curl of E

I'll calculate the curl of **E**, as I did the divergence by studying the simplest possible configuration, a point charge at the origin. In this case

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{\mathbf{r}}$$

what if we calculate the line integral of this field from some point **a** to some other point **b**

$$\int_a^b \mathbf{E} \cdot d\mathbf{l}$$

in spherical coordinates, $d\mathbf{l} = dr\hat{\mathbf{r}} + rd\theta\hat{\theta} + r\sin\theta d\phi\hat{\phi}$, so

$$\mathbf{E} \cdot d\mathbf{l} = \frac{1}{4\pi\varepsilon_0} \frac{1}{r^2} dr$$

therefore

$$\int_a^b \mathbf{E} \cdot d\mathbf{l} = \frac{1}{4\pi\varepsilon_0} \left(\frac{q}{r_a} - \frac{q}{r_b} \right)$$

where r_a is the distance from the origin to point **a** and r_b is the distance to point **b**. The integral around a closed path is evidently zero (for then $r_a = r_b$)

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0$$

and hence applying Stoke's theorem

$$\boxed{\nabla \times \mathbf{E} = 0} \tag{2.5}$$

Equation 2.5 holds for any static charge distribution.

2.3 Electric Potential

The electric field **E** is not just any old vector; it is a special kind of vector function, one whose curl is always zero. There is a theorem that asserts that any vector whose curl is zero is equal to the gradient of some scalar. Because the line integral is indepenent of path, we can define a function

$$V(\mathbf{r}) \equiv -\int_{O}^{r} \mathbf{E} \cdot d\mathbf{l} \tag{2.6}$$

this is called the electric potential. This allows us to write the electric field as

$$\boxed{\mathbf{E} = -\nabla V} \tag{2.7}$$

Equation 2.7 is the differential version of Equation 2.6, it says that the electric field is the gradient of a scalar potential. If you know V, you can easily get **E** by just taking the gradient of V.

2.3.1 Poisson's Equation and Laplace's Equation

We found that the electric field can be written as the gradient of a scalar potential. The question arises: What do the fundamental equations for **E,**

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \qquad \nabla \times \mathbf{E} = 0$$

look like, in terms of V? Well. $\nabla \cdot \mathbf{E} = \nabla \cdot (\nabla V) = -\nabla^2 V$, so apart from the persisting minus sign, the divergence of **E** is the Laplacian of V. Gauss's law then says

$$\boxed{\nabla^2 V = -\frac{\rho}{\varepsilon_0}} \tag{2.8}$$

this is known as **Poisson's equation**. In regions where there is no charge, so that $\rho = 0$, Poisson's equation reduces to **Laplace's equation**

$$\nabla^2 V = 0$$

2.3.2 The Potential of a Localized Charge Distribution

I defined V in terms of **E**. Ordinarily, though it is **E** that we're looking for. The idea is that it might be easier to get V first, and the calculate **E** by taking the gradient. In general, the potential of a point charge q is

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

or for continuous distribution

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{1}{r} dq$$

in particular, for a volume charge

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{r}')}{r} d\tau'$$

this is the equation we are looking for, telling us how to compute V when we know ρ; it is, if you like, the "solution" to Poisson's equation, for a localized charge distribution.

2.3.3 Summary; Electrostatic Boundary Conditions

The electric field always undergoes a discontinuity when you cross a surface charge σ. In fact, it is a simple matter to find the amount by which **E** changes at such a boundary. Gauss's law states that

$$\oint_S \mathbf{E}\cdot d\mathbf{a} = \frac{1}{\varepsilon_0} Q_{\text{enc}} = \frac{1}{\varepsilon_0}\sigma A$$

where A is the area of a pillbox used on the surface. Now the sides of the pillbox contribute nothing to the flux, in the limit as the thickness goes to zero, so we are left with

$$E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{\sigma}{\varepsilon_0}$$

Conclusion: The normal component of **E** is discontinuous by an amount σ/ε_0 at any boundary. In particular, where there is no surface charge, $E^{\perp}$ is contnuous, as for instance at the surface of a uniformly charged solid sphere. The tangential component of **E**, by contrast, is always contnuous.

$$\mathbf{E}^{\parallel}_{\text{above}} = \mathbf{E}^{\parallel}_{\text{below}}$$

the boundary conditions on **E** can be combined into a single formula

$$\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}} = \frac{\sigma}{\varepsilon_0}\hat{\mathbf{n}}$$

where $\hat{\mathbf{n}}$ is a unit vector perpendicular to the surface, pointing from "below" to "above". The potential, meanwhile, is continuous across any boundary, since

$$V_{\text{above}} - V_{\text{below}} = -\int_a^b \mathbf{E}\cdot d\mathbf{l}$$

as the path length shrinks to zero, so too does the integral

$$V_{\text{above}} = V_{\text{below}}$$

However, the gradient of V inherits the discontinuity in **E**; since $\mathbf{E} = -\nabla V$, implies

$$\nabla V_{\text{above}} - \nabla V_{\text{below}} = -\frac{\sigma}{\varepsilon_0}\hat{\mathbf{n}}$$

or more conveniently

$$\frac{\partial V_{\text{above}}}{\partial n} - \frac{\partial V_{\text{below}}}{\partial n} = -\frac{\sigma}{\varepsilon_0}$$

where

$$\frac{\partial V}{\partial n} = \nabla V\cdot\hat{\mathbf{n}}$$

denotes the **normal derivative** of V (that is, the rate of change in the direction perpendicular to the surface).

2.4 Work and Energy in Electrostatics

2.4.1 The Work Done to Move a Charge

Suppose you have a stationary configuration of source charges, and you want to move a test charge Q from point **a** to point **b**. How much work will you have to do? The work is defined as

$$W = \int_a^b \mathbf{F} \cdot d\mathbf{l} = -Q \int_a^b \mathbf{E} \cdot d\mathbf{l} = Q(V(\mathbf{b}) - V(\mathbf{a})]$$

notice that the answer is independent of the path you take from **a** to **b**; in mechanics, then, we would call the electric force "conservative". Dividing through by Q, we have

$$\frac{W}{Q} = V(\mathbf{b}) - V(\mathbf{a})$$

in words, the potential difference between two points is equal to the work-per-unit charge required to carry the particle between the points. In particular, if you want to bring the charge Q in from far away and stick it at a point **r**, the work you must do is

$$W = Q[V(\mathbf{r}) - V(\infty)]$$

so if you have set the reference point out at infinity

$$W = QV(\mathbf{r})$$

2.4.2 The Energy of a Point Charge Distribution

How much work would it take to assemble an entire collection of point charges? Imagine bringing in the charges, one by one, from far away. The work needed to assemble a configuration of point charges is given by

$$\boxed{W = \frac{1}{2} \sum_{i=0}^{n} q_i V(\mathbf{r}_i)} \tag{2.9}$$

this is also the amount of work you'd get back out if you dismantled the system. In the mean time it represents the energy stored in the configuration.

2.4.3 The Energy of a Continuous Charge Distribution

For a volume charge density ρ, Equation 2.9 becomes

$$W = \frac{1}{2} \int \rho V d\tau$$

(The corresponding integrals for line and surface charges would be $\int \lambda V dl$ and $\int \sigma V da$, respectively. This can also be written as

$$\boxed{W = \frac{\varepsilon_0}{2} \int_{\text{space}} E^2 d\tau}$$

The energy can be thought of as being stored in the field or stored in the charge. We can also say

$$\frac{\varepsilon_0}{2} E^2 = \text{energy per unit volume}$$

Because electrostatic energy is quadratic in the field, it does not obey a superposition principle. The energy of a compound system is not the sum of the energies of its parts considered seperately, there are also cross terms

$$W_{\text{tot}} = W_1 + W_2 + \varepsilon_0 \int \mathbf{E}_1 \cdot \mathbf{E}_2 d\tau$$

for example, if you double the charge everywhere, you quadruple the total energy.

2.5 Conductors

2.5.1 Basic Properties

In an **insulator**, such as glass or rubber, each electron is attached to a particular atom. In a metallic **conductor**, by contrats, one or more electrons per atom are free to roam about at will through the material. A perfect conductor would be a material containing an unlimited supply of completely free charges. There are some general rules when dealing with conductors

- **E=0 inside a conductor**
- $\rho = 0$ **inside a conductor**
- **any net charge resides on the surface**
- **A conductor is an equipotential**
- **E is perpendicular to the surface, just outside a conductor**

2.5.2 Surface Charge and the Force on a Conductor

Because the field inside a conductor is zero, boundary conditions requires thatb the field immediately outside is

$$\mathbf{E} = \frac{\sigma}{\varepsilon_0}\hat{\mathbf{n}}$$

consistent with our earlier conclusion that the field is normal to the surface. In terms of potential

$$\sigma = -\varepsilon_0 \frac{\partial V}{\partial n}$$

these equations enable you to calculate the surface charge on a conductor, if you can determine **E** or V. In the presence of an electric field, a surface charge will, naturally experience a force; the force per unit area, **f**, is $\sigma\mathbf{E}$. But there's a problem here, for the electric field is discontinuous at a surface charge, so which value are we supposed to use? The answer is that we should use the average

$$\mathbf{f} = \sigma\mathbf{E}_{\text{average}} = \frac{1}{2}\sigma(\mathbf{E}_{\text{above}} + \mathbf{E}_{\text{below}})$$

the force per unit area is also given as

$$\mathbf{f} = \frac{1}{2\varepsilon_0}\sigma^2\hat{\mathbf{n}}$$

this amounts to an outward **electrostatic pressure** on the surface, tending to draw the conductor into the field, regardless of the sign of σ. Expressing the pressure in terms of the field just outside the surface

$$P = \frac{\varepsilon_0}{2}E^2$$

whic is identical to the energy-per-unit volume stored in the fields.

2.5.3 Capacitors

Suppose we have two conductors, and we put charge $+Q$ on one and $-Q$ the other. Since V is constant over a conductor, we can speak unambiguously of the potential difference between them

$$V = V_{+} - V_{-} = -\int_{-}^{+} \mathbf{E} \cdot d\mathbf{l}$$

We don't know how the charge distributes itself over the two conductors, and calculating the field would be a mess, if their shapes are complicated, but this much we know: **E** is proprtional to Q. Since **E** is proportional to Q, so also is V. The constant of proportionality is called the **capacitance** of the arrangement:

$$\boxed{C = \frac{Q}{V}}$$

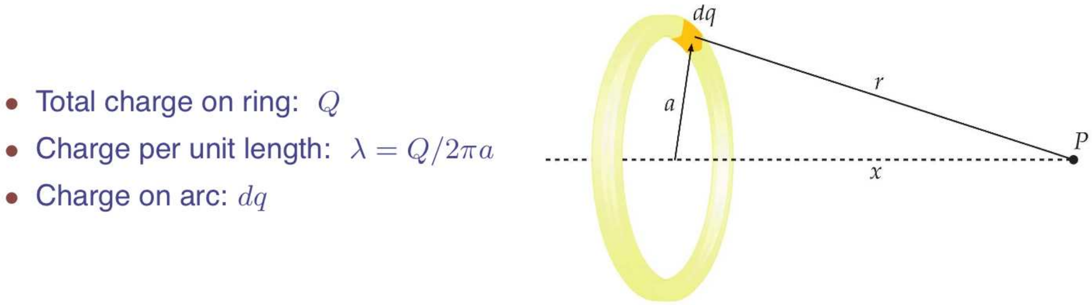
Capacitance is a purely geometrical quantity, determined by the sizes, shapes, and seperation of the two conductors.

2.6 Problems with Solutions

Problem # 1

Electrostatics with cylindrical symmetry

(a). Find the electric potential $V(z)$ a distance z above the center of a circular loop of radius R which carries linear charge density λ. Obtain an expression correct for all values of z (not just $z > 0$).



Total charge on ring: Q
Charge per unit length: $\lambda = Q/2\pi a$
Charge on arc: dq
dq
r
a
P
x

We know that

$$V = -\frac{1}{4\pi\varepsilon_0}\int \frac{dq}{r}$$

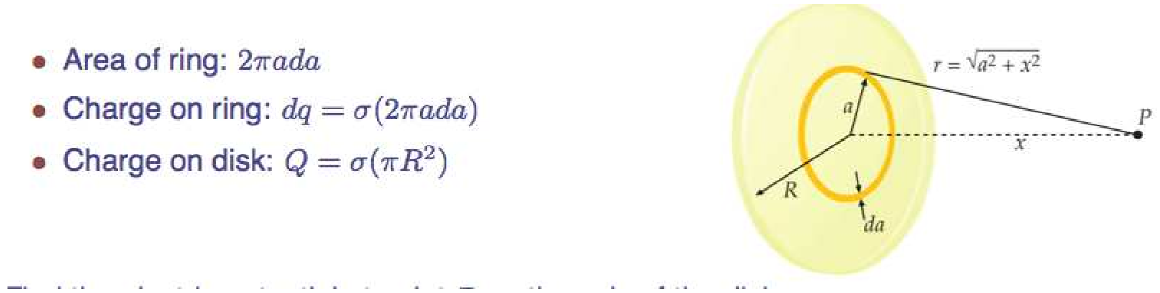
since we know that dq is given by

$$dq = \lambda ds = \lambda R d\phi \quad r = \sqrt{R^2 + z^2}$$

thus the above expression for the potential is

$$V = -\frac{1}{4\pi\varepsilon_0}\int_0^{2\pi} \frac{\lambda R d\phi}{\sqrt{R^2+z^2}} = \boxed{-\frac{\lambda}{2\varepsilon_0}\frac{R}{\sqrt{R^2+z^2}}}$$

(b). Find the electric potential a distance z above the center of a circular disk of radius R, which carries a uniform surface charge σ.



- Area of ring: $2\pi a da$
- Charge on ring: $dq = \sigma(2\pi a da)$
- Charge on disk: $Q = \sigma(\pi R^2)$

We know that

$$dq = \sigma da = \sigma 2\pi r dr$$

putting this into the expression for the potential we find

$$V = -\frac{1}{4\pi\varepsilon_0}\int \frac{dq}{r} = V = -\frac{1}{4\pi\varepsilon_0}\int_0^R \frac{2\pi\sigma r}{\sqrt{z^2+r^2}}dr$$

using a u substitution we find the solution to the integral to which is given by

$$u = z^2 + r^2 \quad du = 2rdr \quad (r = 0 : u \to z^2, r = R : u \to z^2 + R^2)$$

which gives us

$$V = -\frac{\sigma}{4\varepsilon_0}\int_{z^2}^{z^2+R^2} u^{-1/2}du = \boxed{-\frac{\sigma}{2\varepsilon_0}\left[\sqrt{z^2+R^2} - z\right]}$$

(b). Using the potential and the symmetry of the problem, determine the electric field a distance z above the center of the circular disk for part **b.** Check that your answer makes sense: What do you get for $|z| \gg R$ and for $z \gg R$: Show also that he magnitude of the field at z is proprtional to the solid angle subtended by the disk as seen from the point z.

We know that the electric field can be defined as

$$\mathbf{E} = -\nabla V = \frac{\sigma}{2\varepsilon_0}\frac{\partial}{\partial z}\left[\sqrt{z^2+R^2} - z^2\right]\hat{\mathbf{z}} = \boxed{\frac{\sigma}{2\varepsilon_0}\left[\frac{z}{\sqrt{z^2+R^2}} - 1\right]\hat{\mathbf{z}}}$$

For $R \ll z$ we can use a Taylor expansion

$$(1+x)^n \approx 1 + nx$$

so we find

$$\mathbf{E} = \frac{\sigma}{2\varepsilon_0}\left[\frac{z}{\sqrt{z^2+R^2}} - 1\right]\hat{\mathbf{z}} = \frac{\sigma}{2\varepsilon}\left[\left(1 + \frac{R^2}{z^2}\right)^{-1/2} - 1\right]\hat{\mathbf{z}} \approx -\frac{\sigma}{4\varepsilon_0}\frac{R^2}{z^2}\hat{\mathbf{z}}$$

and we know that

$$q = \sigma\pi R^2 \quad z = R$$

thus at $R \ll z$ we should see a point charge i.e

$$\mathbf{E} = -\frac{q}{4\pi\varepsilon_0 R^2}\hat{\mathbf{z}}$$

For $R \gg z$ we can use a Taylor expansion

$$(1+x)^n \approx 1+nx$$

so we find

$$\mathbf{E} = \frac{\sigma}{2\varepsilon_0}\left[\frac{z}{\sqrt{z^2+R^2}}-1\right]\hat{\mathbf{z}} = \left[\frac{z}{R}\left(1+\frac{z^2}{R^2}\right)^{-1/2}-1\right]\frac{\sigma}{2\varepsilon}\hat{\mathbf{z}} \approx -\frac{\sigma}{2\varepsilon_0}\left[\frac{1}{2}\frac{z^3}{R^3}-\frac{z}{R}+1\right]\hat{\mathbf{z}}$$

this is only for $R \gg z$.

To show also that the magnitude of the field at z is proprtional to the solid angle subtended by the disk as seen from the point z. We will say

$$\Omega = \int_0^{2\pi} d\phi \int_0^{\theta} \sin\theta d\theta = -2\pi\cos\theta$$

but the limits of θ are given by

$$0 \text{ and } \theta = \cos^{-1}\left(\frac{z}{\sqrt{z^2+R^2}}\right)$$

which gives us

$$\Omega = 2\pi z\left[\frac{z}{\sqrt{z^2+R^2}}-1\right]$$

and

$$\frac{\mathbf{E}}{\Omega} = \frac{\sigma}{4\pi\varepsilon_0} = \text{constant}$$

thus

$$\mathbf{E} = k\Omega$$

Problem # 2

Two spheres, each of radius R carrying uniform charge densitys $+\rho$ and $-\rho$, respectively, are placed so that they partialy overlap. Call the vector from the positive center to the negative center $\mathbf{d}$. Show that the field in the region of overlap is constant, and find its value.

We can solve this problem by finding the electric fields for 1 sphere and adding the spheres using superposition. It is well known that the electric field of a uniform sphere is given by

$$\oint E\cdot d\mathbf{a} = \frac{q_{tot}}{\varepsilon_0} \quad q_{tot} = \rho(r)V = \frac{4}{3}\rho(r)\pi r^3$$

since the electric field is constant we find

$$\mathbf{E}A = \frac{4}{3}\rho(r)\pi r^3\frac{1}{\varepsilon_0} \quad \mathbf{E} = \frac{\rho(r)r}{3\varepsilon_0}\hat{\mathbf{r}}$$

so we find that the two spheres are given by

$$\mathbf{E}_+ = \frac{\rho(r)r_+}{3\varepsilon_0}\hat{\mathbf{r}} \quad \mathbf{E}_- = -\frac{\rho(r)r_-}{3\varepsilon_0}\hat{\mathbf{r}}$$

and so the total electric field is just the sum of these two (superposition), we also know that the vectors are pointing towards each other and that the vertical components of the vectors cancel by symmetry. We can find the total electric field as

$$\mathbf{E}_t = \mathbf{E}_+ + \mathbf{E}_- = \left[\frac{\rho(r) r_+}{3\varepsilon_0} - \frac{\rho(r) r_-}{3\varepsilon_0}\right]\hat{\mathbf{r}} = \frac{\rho(r)}{3\varepsilon_0}[r_+ - r_-]\hat{\mathbf{r}}$$

but since we know that the total distance between this two vectors will always be the same we can write the r_- vector as $-r_-$which would give you just give you **d** (which is constant) and thus your total electric field would be constant. i.e

$$\mathbf{d} = r_+ + r_- = r_+ - r_-$$

Problem # 3

Consider the so-called "screened Coulomb potential" of a point charge q that arises, for example, in plasma physics

$$V(r) = \frac{q}{4\pi\varepsilon_0}\frac{e^{-r/\lambda}}{r}$$

where λ is a constant (called the screening length)

(a). Determine the charge distribution $\rho(r)$ that produces this potential. Sketch this function in a manner that clearly describes all of its characteristics (i.e what's the best way of representing this three-dimensional charge distribution? Use it and explain what your potting.

We can solve this problem two ways, you can take the Laplacian of a Laplacian or just use the following identities

$$\nabla^2 V = \nabla \cdot (\nabla(fg)) = \nabla \cdot (f\nabla g + g\nabla f)$$

and we also know the following relationships

$$\mathbf{E} = -\nabla V \quad \nabla \cdot \mathbf{E} = \nabla \cdot (-\nabla V) = \frac{\rho(r)}{\varepsilon_0}$$

thus we know that

$$\rho(r) = \varepsilon_0(\nabla \cdot \mathbf{E}) = \varepsilon_0(\nabla \cdot (-\nabla V))$$

so we need to find what the divergence of the electric field is, by finding the gradient of the potential first and taking the divergence of this

$$\nabla V = \frac{\partial}{\partial r}\left(\frac{q}{4\pi\varepsilon_0}\frac{e^{-r/\lambda}}{r}\right) = \frac{q}{4\pi\varepsilon_0}\frac{\partial}{\partial r}\left(\frac{e^{-r/\lambda}}{r}\right) = -\frac{e^{-r/\lambda}}{r^2}\left[1 + \frac{r}{\lambda}\right]\hat{\mathbf{r}}$$

thus we find

$$\mathbf{E} = \frac{e^{-r/\lambda}}{r^2}\left[1 + \frac{r}{\lambda}\right]\hat{\mathbf{r}}$$

letting

$$f = e^{-r/\lambda}\left[1 + \frac{r}{\lambda}\right] \quad \text{and} \quad g = \frac{\hat{\mathbf{r}}}{r^2}$$

we find

$$\nabla \cdot \mathbf{E} = e^{-r/\lambda}\left(1 + \frac{r}{\lambda}\right)\nabla \cdot \frac{\hat{\mathbf{r}}}{r^2} + \frac{\hat{\mathbf{r}}}{r^2}\nabla e^{-r/\lambda}\left(1 + \frac{r}{\lambda}\right)$$

but we know from Griffith page 50 that

$$\nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2}\right) = 4\pi\delta^3(r)$$

so we find

$$\begin{aligned}\nabla \cdot \mathbf{E} &= e^{-r/\lambda}\left(1+\frac{r}{\lambda}\right)4\pi\delta^3(r) + \frac{\hat{\mathbf{r}}}{r^2}\nabla e^{-r/\lambda}\left(1+\frac{r}{\lambda}\right) \\ &= e^{-r/\lambda}\left(1+\frac{r}{\lambda}\right)4\pi\delta^3(r) + \frac{1}{r^2}\frac{\partial}{\partial r}\left(e^{-r/\lambda}\left(1+\frac{r}{\lambda}\right)\right) \\ &= \frac{q}{4\pi\varepsilon_0}\left[e^{-r/\lambda}\left(1+\frac{r}{\lambda}\right)4\pi\delta^3(r) - \frac{e^{-r/\lambda}}{r\lambda^2}\right]\end{aligned}$$

thus the charge density is given by

$$\rho(r) = \boxed{\frac{q}{4\pi}\left[e^{-r/\lambda}\left(1+\frac{r}{\lambda}\right)4\pi\delta^3(r) - \frac{e^{-r/\lambda}}{r\lambda^2}\right]}$$

The plot is given by

where the plot is a countour map that describes the distribution with respect to the radius.

(b). Show by explicit calculation over $\rho(r)$ that the net charge represented by this distribution is zero. Explain how this observation is easily derived using the integral form of Gauss' law.

the total charge Q can be found by

$$Q_{net} = \int_V \rho(r)d\tau = \frac{q}{4\pi}\left[\int_{space} e^{-r/\lambda}\left(1+\frac{r}{\lambda}\right)4\pi\delta^3(r)d\tau - \int_{space}\frac{e^{-r/\lambda}}{r\lambda^2}d\tau\right]$$

which gives us

$$\int_V \rho(r)d\tau = \frac{q}{4\pi}\left[4\pi - 4\pi\int_0^\infty \frac{re^{-r/\lambda}}{\lambda^2}dr\right]$$

where the term on the right hand side is just the Gamma function and the delta function integral is just 4π by definition. So we find that the total charge is zero

$$\int_0^\infty \frac{re^{-r/\lambda}}{\lambda^2}dr = 1$$

$$Q(tot) = \int_{space} \rho(r) d\tau = \frac{q}{4\pi}\left[4\pi - 4\pi\right] = 0$$

this tedious calculation could have been solved by using the divergence theorem and Gauss's law. i.e

$$\int_V (\nabla \cdot \mathbf{E}) dV = \oint_s \mathbf{E} \cdot d\mathbf{a} = \frac{q_{enc}}{\varepsilon_0}$$

but we know t

$$\int_V (\nabla \cdot \mathbf{E}) dV \quad = \quad \int \left(-\frac{\rho(r)}{\varepsilon_0}\right) dV = 0 = \oint \mathbf{E} \cdot d\mathbf{a} = \frac{q_{enc}}{\varepsilon_0}$$

Since we now know that we have a +q charge due to the delta function and a -q charge from the integral of the potential contained within a surface S then the surface has the same number of field lines coming in that are going out and thus the surface integral goes to zero because the divergence goes to zero thus implying that q_{enc} is zero.

Problem # 4

(a). Show that the electrostatic energy of a uniformly charged solid sphere, with total charge Q and radius R is $3Q^2/(20\pi\varepsilon_0 R)$.

It is assumed that we know what the electric field is in the inside of the sphere and on the outside of the sphere is. i.e

$$\mathbf{E}_{in} = \frac{q}{4\pi\varepsilon_0 R^3} r\hat{\mathbf{r}} \qquad \mathbf{E}_{out} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}}$$

and using the following expression for the electrostatic energy

$$W = \frac{\varepsilon_0}{2}\int_{space} E^2 d\tau$$

thus we must consider two integrals, the contribution from the inside and the contribution from the outside. Knowing the limits of integration we find

$$\begin{aligned} W &= \frac{q^2}{8\pi\varepsilon_0}\left[\int_0^R \frac{r^4}{R^6} dr + \int_R^\infty \frac{1}{r^2} dr\right] = \frac{q^2}{8\pi\varepsilon_0}\left[\frac{1}{5R} + \frac{1}{R}\right] \\ W &= \boxed{\frac{3q^2}{20\pi\varepsilon_0 R}} \end{aligned}$$

(b). Use the result above to compute the electrostatic energy of an atomic nucleus with Z protons and a total of A nucleons, using an approximation for the nuclear radius as $R = (1.2 \times 10^{-15}\text{m})A^{1/3}$. Give your results in units of MeV times $Z^2/A^{1/3}$

this is just a plug and chug problem, figure out your constants and plug away, i.e

$$W = \frac{3(Zq)^2}{20\pi\varepsilon_0 (1.2 \times 10^{-15}\text{m}) A^{1/3}} \boxed{0.720\text{MeV}\frac{Z^2}{A^{1/3}}}$$

(c). Calculate the change of electrostatic energy when a uranium nucleus undergoes fission.

We know that U^{235} decays as

$$U^{235} \rightarrow ^{92}_{36}Kr + ^{141}_{56}Ba + 3n$$

thus we know that

$$\Delta W = W_{reac} - W_u = |(W_{kr} + W_B) - W_u$$

thus

$$\Delta W = 0.720\text{MeV}\left[\left(\frac{36^2}{92^{1/3}} + \frac{56^2}{141^{1/3}}\right) - \frac{92^2}{235^{1/3}}\right] = -347.02\text{MeV}$$

which is a large amount of liberated energy.

Problem # 5
Evaluate the following integrals

(a).

$$\int_{space} (r^2 + \mathbf{r}\cdot\mathbf{a} + a^2)\delta^3(\mathbf{r}-\mathbf{a})d\tau$$

where **a** is a fixed vector and a is its magnitude.

We know that the delta function is zero everywhere except at $\mathbf{r} = \mathbf{a}$ thus we must evaluate this integral at **a** i.e

$$\int f(\mathbf{r})\delta^3(\mathbf{r}-\mathbf{a})d\tau = f(\mathbf{a})$$

thus this yields

$$\int_{space} (r^2 + \mathbf{r}\cdot\mathbf{a} + a^2)\delta^3(\mathbf{r}-\mathbf{a})d\tau = \boxed{3a^2}$$

(b).

$$\int_V |\mathbf{r}-\mathbf{b}|^2\delta^3(5\mathbf{r})d\tau$$

where V is a cube of side 2, centered on the origin, and $\mathbf{b} = 4\hat{\mathbf{y}} + 3\hat{\mathbf{z}}$.

We know that we must evaluate this function at $\mathbf{r} = 0$, which yields

$$\int_V |\mathbf{r}-\mathbf{b}|^2\delta^3(5\mathbf{r})d\tau = |-\mathbf{b}|^2 = \boxed{25}$$

(c).

$$\int_V (r^4 + r^2(\mathbf{r}\cdot\mathbf{c}) + c^4)\delta^3(\mathbf{r}-\mathbf{c})d\tau$$

where V is a sphere of radius 6 about the origin, and $\mathbf{c} = 5\hat{\mathbf{x}} + 3\hat{\mathbf{y}} + 2\hat{\mathbf{z}}$.

We know that the delta function is zero everywhere excpet for where it is defined, so this means if the limits of the integral are not within the specified value at which the delta function is equal to 1 then the integral is zero.

$$c = \sqrt{5^2 + 3^2 + 2^2} = \sqrt{38} = 6.16 \quad \Rightarrow 6.16 > 6.0$$

thus this implies that the magnitude of **c** is outside of the range and so

$$\int_V (r^4 + r^2(\mathbf{r}\cdot\mathbf{c}) + c^4)\delta^3(\mathbf{r}-\mathbf{c})d\tau = \boxed{0}$$

(d).

$$\int_V \mathbf{r}\cdot(\mathbf{d}-\mathbf{r})\delta^3(\mathbf{e}-\mathbf{r})d\tau$$

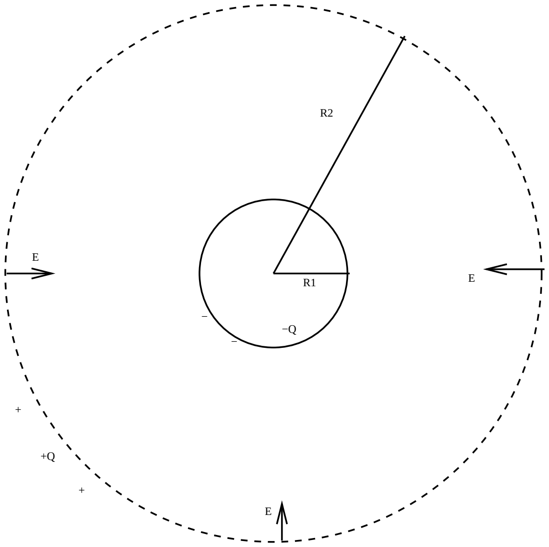
where $\mathbf{d}=(1,2,3)$,and $\mathbf{e}=(3,2,1)$, and V is a sphere of radius 1.5 centered at (2,2,2).

From our experience with delta function we now know that this will be given by

$$\int_V \mathbf{r}\cdot(\mathbf{d}-\mathbf{r})\delta^3(\mathbf{e}-\mathbf{r})d\tau = \mathbf{e}\cdot(\mathbf{d}-\mathbf{e}) = \boxed{-4}$$

Problem # 6

The farad is actually an enormous unit of capacitance. To illustrate this, treat the Earth as a conducting sphere and find its capacitance.



The above figure describes the earth capacitance problem, Where R_1 is the radius of the earth and also $R_2 \gg R_1$ and also a positive charge Q on the outside surface and a negative charge Q for the earth.

We know that the capacitance is defined as

$$C = \frac{Q}{V}$$

where Q is the charge and V is the potential. The potential is given as

$$V = -\int_{R_1}^{R_2} \mathbf{E}\cdot d\mathbf{r} = -\frac{1}{4\pi\varepsilon_0}\int_{R_1}^{R_2} \frac{Q}{r^2}(\cos\pi)dr = -\frac{1}{4\pi\varepsilon_0}\left[\frac{1}{R_2}-\frac{1}{R_1}\right]$$

because the electric field points in the $-\hat{\mathbf{r}}$ direction we must consider the $\cos\pi$ term. Thus we find

$$V = -\frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{R_2}-\frac{1}{R_1}\right]$$

and so the capacitance is given by

$$C = -\frac{R_1R_2 4\pi\varepsilon_0}{R_2-R_1} \qquad R_2 \gg R_1 \quad R_1 = 6,378\,\text{km}$$

so we find

$$C = R_1 4\pi\varepsilon_0 = \boxed{700\ \mu F}$$

Problem # 7

Consider a system of n conductors that are prepared with total charges of Q_i , with i∈{1, 2, 3, . . . n} labeling the conductors. The surface charge densities on each of the conductors are described by the functions σ_i , for which, clearly

$$Q_i = \int_{S_i} \sigma_i da_i$$

with the integral being over the surface S_i of conductor i.

(a). Find an expression for the potential on conductor i. This you may obtain as usual by summing over the contributions to the potential from all the surface charges on all the conductors (including the $i-th$ conductor).

Since we know that there i_{th} and j_{th} conductors, to find the potential of the i_{th} conducter from all the j_{th} conductors, we must sum over all the j_{th} conducters contribution to the potential, i.e

$$V_i = \frac{1}{4\pi\varepsilon_0} \int_{S_j} \frac{Q_j}{r_{ij}} = \sum_{j=1} \int_{S_j} \frac{\sigma_j da_j}{r_{ij}}$$

where r_{ij} is the distance from the i_{th} conducter to the j_{th} conductor.

(b). Let us define a coefficient p_{ij} as

$$p_{ij} = \frac{1}{4\pi\varepsilon_0 S_j} \int_{S_j} \frac{f_i da_j}{r_{ij}}$$

Here, r_{ij} denotes the distance from one point on conductor i to the point on conductor j at which the integrand is evaluated (as we integrate over the surface of j). In this integral, we make us of a relation

$$\sigma_i = \langle\sigma_j\rangle f_j = \frac{Q_j}{S_j} f_j$$

in which S_j is now the area of the j_{th} conductor. The dimensionless function f_j describes how the actual charge density differs from the average. With this definition, show that the following set of equations hold:

$$\begin{aligned} V_1 &= p_{11}Q_1 + p_{12}Q_2 + \ldots + p_{1n}Q_n \\ V_2 &= p_{21}Q_1 + p_{22}Q_2 + \ldots + P_{2n}Q_n \\ &. \\ &. \\ V_n &= p_{n1}Q_1 + p_{n2}Q_2 + \ldots + P_{nnQ_n} \end{aligned}$$

The coefficients p_{ij} are known as coefficients of potential. They express how the potential on conductor i will vary (linearly) upon changing the charge Q_j on conductor j . We note here that the coefficients no longer contain any reference to the specific potentials or charges placed on the conductors - they simply reflect geometric properties of the conductors and their placements.

(c). By considering the quantity $p_{ij}Q_iQ_j$, show that $p_{ij} = p_{ji}$. This nice relation expresses the following fact: if a charge Q on conductor i brings that conductor to a potential V , then the same charge Q placed on conductor i would bring conductor j to the same potential V.

This can be shown by

$$p_{ij}Q_iQ_j = \frac{Q_iQ_j}{4\pi\varepsilon_0 S_j}\int_{S_j}\frac{f_j da_j}{r_{ij}} \qquad \frac{Q_jQ_i}{4\pi\varepsilon_0 S_i}\int_{S_i}\frac{f_i da_i}{r_{ji}} = p_{ji}Q_jQ_i$$

and using the fact that

$$\sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}B_{ji} = \sum_{j=1}^{n}\sum_{i=1}^{n} B_{ij}A_{ji}$$

This will shows the symmetry of the problem, i.e whatever conductor i sees in the presence of conductor j of charge Q will be exactly the same if conductor i of charge Q as seen by conductor j.

(d). Consider a situation with only two conductors that are used as a capacitor. What is the capacitance for this system expressed in terms of the coefficients of potential p_{11} , p_{12} and p_{22} ?

if we know that

$$\begin{aligned} V_1 &= p_{11}Q_1 + p_{12}Q_2 \\ V_2 &= p_{21}Q_1 + p_{22}Q_2 \end{aligned}$$

we also know that

$$p_{12} = p_{21}$$

thus we can find the potential by using

$$\Delta V = V_2 - V_1 = (p_{12} - p_{11})Q_1 + (p_{22} - p_{12})Q_2$$

and we we assume that $Q_1 = -Q_2$ we find

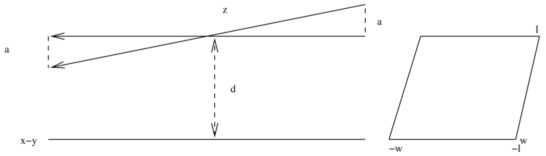
$$C = \frac{Q_1}{\Delta V} = \frac{1}{p_{11} + p_{12} + p_{22}}$$

Problem # 8

Suppose that a parallel plate ca-pacitor has rectangular plates but the plates are not exactly parallel. The separation at one edge is $d - a$ and $d + a$ at the other edge, where $a \ll d$. Show that the capacitance is given approximately by

$$C \simeq \frac{\varepsilon_0 A}{d}\left(1 + \frac{a^2}{3d^2}\right)$$

where A is the area of the plate. A figure is given by



Thus we know from the diagram

$$A = 4lw$$

and knowing from Gauss's law

$$\oint \mathbf{E} \cdot d\mathbf{a} = \frac{\sigma dA}{2\varepsilon_0}$$

which give us

$$\mathbf{E}dA = \frac{\sigma dA}{2\varepsilon_0} \quad \Rightarrow \mathbf{E} = \frac{\sigma}{2\varepsilon_0}(-\hat{\mathbf{z}})$$

to find the potential we must use

$$V = -\int \mathbf{E} \cdot dl$$

and knowing that

$$d\mathbf{l} = -\hat{\mathbf{z}}ds$$

where ds is in the direction of the field. Thus we find the potential to be given by

$$V = -\int \frac{\sigma}{2\varepsilon_0}(\hat{\mathbf{z}}) \cdot (-\hat{\mathbf{z}}ds) = \frac{\sigma}{2\varepsilon_0}S$$

which S is the the distance between the plane that is a function of x which is given by the equation of the line in the form

$$S = d + \frac{a}{w}x$$

thus we now know that the capacitance is given by and $q_{enc} = \sigma dA$

$$C = \int_{-w}^{w} \frac{2\varepsilon_0 \sigma dA}{\sigma\left(d + \frac{a}{w}x\right)} = \int_{-w}^{w} 2\varepsilon_0 l \int_{-w}^{w} \frac{dx}{\left(d + \frac{a}{w}\right)}$$

using a u substitution, in which we will just qoute the solution. We know how to do u substitutions at this level

$$C = 2\varepsilon_0 \frac{lw}{a}\left[\ln\left[d\left(1 + \frac{a}{d}\right)\right] - \ln\left[d\left(1 - \frac{a}{d}\right)\right]\right]$$

which if we do a bit of algebra with a knowledge of how the natural log functions work along with the Taylor expansion for the natural log function which is

$$\ln(x+1) \approx x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

we find

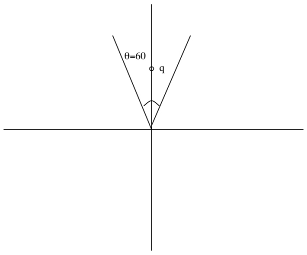
$$C = 2\varepsilon_0 \frac{lw}{a}\left[\frac{a}{2} - \frac{a^2}{2d^2} + \frac{a^3}{3d^3} - \left[-\left[-\frac{a}{d} - \frac{a^2}{2d^2} - \frac{a^3}{3d^3}\right]\right]\right]$$

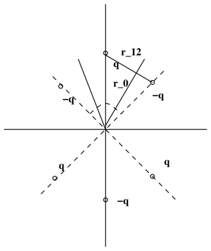
the even terms cancel and thus to third order we find, with a bit of algebra

$$C = 2lw\varepsilon_0 \frac{1}{a}\left[\frac{2a}{d} + \frac{a^3}{3d^3}\right] = \boxed{4lw\varepsilon_0 \frac{1}{d}\left[1 + \frac{a^2}{3d^2}\right] = \frac{A\varepsilon_0}{d}\left[1 + \frac{a^2}{3d^2}\right]}$$

Problem # 9

(a) Two grounded conducting planes meet at an angle of $\theta = 60^0$ at the origin. A point charge q is a distance r_0 from the origin along their angular bisector. What image charges are needed to satisfy the boundary conditions? (Hint: their arrangement is going to be highly symmetric)





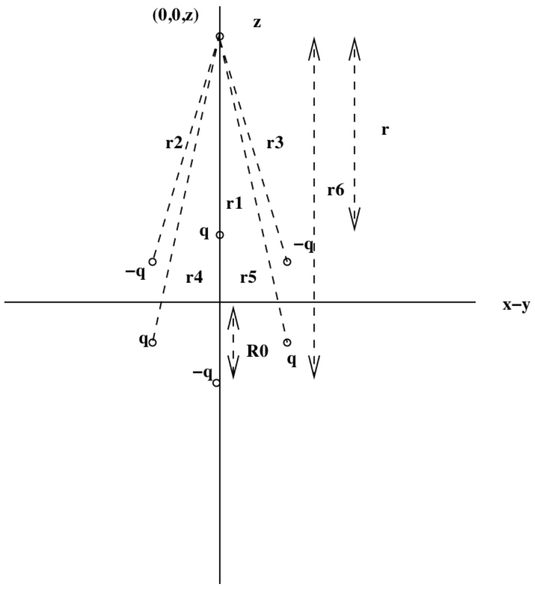
We can see from the diagram will require 6 total charges and 5 image charges. This is necessary to meet the boundary condition implied by the use of images. We can see that there is strong dependence in the angle between the plates. You must balance the charges. where the total charge is zero.

What is the r depedence of the potential along the angular bisector for $r \gg r_0$?

We can find the r dependence is given by

$$V = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\mathbf{r_1}} - \frac{1}{\mathbf{r_2}} - \frac{1}{\mathbf{r_3}} + \frac{1}{\mathbf{r_4}} + \frac{1}{\mathbf{r_5}} - \frac{1}{\mathbf{r_6}}\right]$$

if we pick an arbitrary point $(0,0,z)$ we can find a relationship for the distance between these charges and that point. The diagram is given by



knowing that the distance between each pair of charges is the same because it is a equilatiral triangle and by symmetry we find

$$r_4 = r_5 \quad \text{and also} \quad r_2 = r_3$$

and also from the diagram we can see

$$r_1 = r - R_0 \qquad r_6 = r + R_0$$

Thus we find that the potential is given by

$$V = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r - R_0} - \frac{2}{r_2} + \frac{2}{r_4} - \frac{1}{r + R_0}\right]$$

combining the first term and the last term gives us

$$V = \frac{q}{4\pi\varepsilon_0}\left[\frac{2R_0}{r^2+R_0^2} - \frac{2}{r_2} + \frac{2}{r_4}\right]$$

From the diagram we can also infer that

$$r_2^2 = \frac{3}{4}R_0^2 + \left(r - \frac{R_0}{2}\right)^2$$

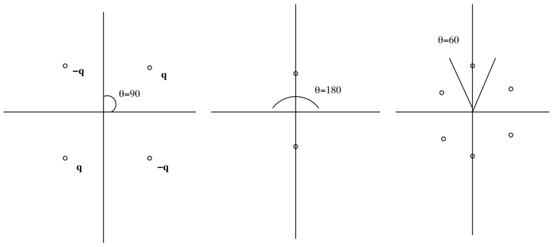
and by symmetry we know that

$$r_4^2 = \frac{3}{4}R_0^2 + \left(r + \frac{R_0}{2}\right)^2$$

thus we find that the r dependence on the potential is given by

$$V = \frac{q}{4\pi\varepsilon_0}\left[\frac{2R_0}{r^2+R_0^2} - \frac{2}{\sqrt{\frac{3}{4}R_0^2 + \left(r - \frac{R_0}{2}\right)^2}} + \frac{2}{\sqrt{\frac{3}{4}R_0^2 + \left(r + \frac{R_0}{2}\right)^2}}\right]$$

(b) Repeating the above problem but for different angles θ between the grounded plates. Show that if $\theta = 2\pi/n$, where n is any even integer, then $n-1$ image charges are required to ground both sides of the wedge. Show that if, however, n is an odd integer (e.g. $n = 3$) then an image solution cannot be obtained.



We can see that there is a dependence on n by

$$\frac{360}{90} = 4\ (4-1) = 3 \quad \frac{360}{180} = 2\ (2-1) = 1 \quad \frac{360}{60} = 6\ (6-1) = 5$$

from these three examples we can see that

$$\frac{2\pi}{\theta} = n \text{ total number of charges}$$

and

$$n-1 \text{ image charges}$$

we can also see that θ must be even to allow an even number of charges that to be used in order to satisfy the boundary conditions.

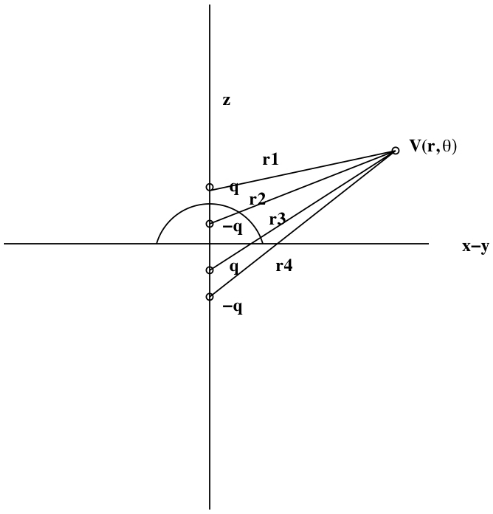
Problem # 10

A conductor is formed with a grounded conducting plane spanning the $x-y$ axis ($z = 0$). Upon it is attached a hemispherical conductor of radius R_0 that extends into the half-plane $z > 0$ and whose center is at the origin. A charge q is placed at the point $(x, y, z) = (0, 0, z)$ with $z > R_0$.

(a). What is the electrostatic potential $V(r,\theta)$ with r and θ being conventionally defined spherical coordinates (by symmetry, the potential must be independent of the azimuthal angle ϕ)? Hint: there will be three image charges. Answer:

$$1-\frac{z^2-R_0^2}{z\sqrt{z_0^2+R_0^2}}$$

A diagram is given by



Thus we know that

$$V(r,\theta)=\frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\mathbf{r_1}}-\frac{1}{\mathbf{r}_2}+\frac{1}{\mathbf{r_3}}-\frac{1}{\mathbf{r_4}}\right]$$

Using geometry and symmetry, along with the the example in Griffiths's we find that the relationships for the all the radius are given as

$$\begin{aligned}\mathbf{r_1}^2 &= r^2\sin^2\theta+(r\cos\theta-z)^2\\ \mathbf{r_2}^2 &= r^2\sin^2\theta+\left(r\cos\theta-\frac{R_0^2}{z}\right)^2\end{aligned}$$

and by symmetry we find that the other two radius are given by

$$\begin{aligned}\mathbf{r_3}^2 &= r^2\sin^2\theta+\left(r\cos\theta+\frac{R_0^2}{z}\right)^2\\ \mathbf{r_4}^2 &= r^2\sin^2\theta+(r\cos\theta+z)^2\end{aligned}$$

so we know that the potential is given by after cleaning it up a bit

$$V(r,\theta)=\frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{r^2+z^2-2rz\cos\theta}}-\frac{R_0}{z\sqrt{r^2-\frac{2R_0^2r\cos\theta}{z}+\frac{R_0^4}{z^2}}}+\frac{R_0}{z\sqrt{r^2+\frac{2R_0^2r\cos\theta}{z}+\frac{R_0^4}{z^2}}}-\frac{1}{\sqrt{r^2+z^2+2rz\cos\theta}}\right]$$

and we know that the total charge distribution induced on the plane is given by

$$\sigma_p=-\varepsilon_0\frac{1}{r}\frac{dV}{d\theta}$$

and so we find that the derivative of the potential with respect to θ is given as

$$\begin{aligned}\frac{dV}{d\theta} &= q\left[\frac{rR_0^3\sin\theta}{z^2\left(r^2-\frac{2R_0^2r\cos\theta}{z}+\frac{R_0^4}{z^2}\right)^{3/2}}+\frac{rR_0^3\sin\theta}{z^2\left(r^2+\frac{2R_0^2r\cos\theta}{z}+\frac{R_0^4}{z^2}\right)^{3/2}}-\frac{rz\sin\theta}{\left(r^2+z^2-2rz\cos\theta\right)^{3/2}}\right]\\ &- q\left[\frac{rz\sin\theta}{\left(r^2+z^2-2rz\cos\theta\right)^{3/2}}\right]\end{aligned}$$

thus we know that

$$\begin{aligned}\sigma_p &= -\frac{q}{4\pi}\left[\frac{R_0^3\sin\theta}{z^2\left(r^2-\frac{2R_0^2r\cos\theta}{z}+\frac{R_0^4}{z^2}\right)^{3/2}}+\frac{R_0^3\sin\theta}{z^2\left(r^2+\frac{2R_0^2r\cos\theta}{z}+\frac{R_0^4}{z^2}\right)^{3/2}}-\frac{z\sin\theta}{\left(r^2+z^2-2rz\cos\theta\right)^{3/2}}\right]\\ &- \left(\frac{q}{4\pi}\right)\left[\frac{z\sin\theta}{\left(r^2+z^2-2rz\cos\theta\right)^{3/2}}\right]\end{aligned}$$

and since we are integrating over the plane this means $\theta=\frac{\pi}{2}$ and $d\mathbf{a}=r\sin\theta d\phi dr$ and

$$q_p=\int_{space}\sigma d\mathbf{a}$$

which will give us

$$q_p=\frac{q}{4\pi}\int_0^{2\pi}d\phi\int_{R_0}^{\infty}r\left[\frac{R_0^3}{z^2\left(r^2+\frac{R_0^4}{z^2}\right)^{3/2}}+\frac{R_0^3}{z^2\left(r^2+\frac{R_0^4}{z^2}\right)^{3/2}}-\frac{z}{(r^2+z^2)^{3/2}}-\frac{z}{(r^2+z^2)^{3/2}}\right]$$

which becomes

$$q_p=-\frac{q}{2}\int_{R_0}^{\infty}\frac{R_0^3rdr}{z^2\left(r^2+\frac{R_0^4}{z^2}\right)^{3/2}}+\frac{q}{2}\int_{R_0}^{\infty}\frac{zrdr}{(r^2+z^2)^{3/2}}$$

solving this integral using u substitution yields

$$q_p=-q\left[\frac{z}{2}(-2)\frac{-1}{\sqrt{R_0^2+z^2}}-\frac{R_0^3(-2)}{2z^2}\frac{-1}{\sqrt{R_0^2+\frac{R_0^4}{z^2}}}\right]$$

simpyfying this term yields

$$q_p=-q\left[\frac{z}{\sqrt{R_0^2+z^2}}-\frac{R_0^2}{z^2}\frac{z}{\sqrt{z^2+R_0^2}}\right]=-q\frac{z^2+R_0^2}{z\sqrt{z^2+R_0^2}}$$

and knowing that

$$q_p+q_h=-q \quad \Rightarrow \frac{q_h}{-q}=1-\frac{q_p}{-q}$$

thus

$$\frac{q_h}{-q}=\boxed{1-\frac{z^2+R_0^2}{z\sqrt{z^2+R_0^2}}}$$

Chapter 3

Special Techniques

3.1 Laplace's Equation

The primary task of electrostatics is to find the electric field of a given stationary charge distribution. In principle, this purpose is accomplished by Coulomb's law, in the form of Equation 2.3

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\hat{\mathbf{r}}}{r^2}\rho(\mathbf{r}')d\tau'$$

Unfortunately, integrals of this type can be difficult to calculate for any but the simplest charge configurations. We can attempt to solve for the potential using

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{1}{r}\rho(\mathbf{r}')d\tau' \tag{3.1}$$

Still, even this integral is often too tough to handle analytically. In such cases it is fruitful to recast the problem in differential, using Poisson's equation

$$\nabla^2 V = \frac{1}{\varepsilon_0}\rho$$

which, together with appropriate boundary conditions, is equivalent to Equation 3.1. Very often in fact, we are interested in finding the potential in a region where $\rho = 0$. There may be plenty of charge elsewhere, but we're confining our attention to places where there is no charge. In this case Poisson's equation reduces to Laplace's equation

$$\nabla^2 V = 0$$

or, written in cartesian coordinates

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

this formula is so fundamental to the subject that one might almost say electrostatics is the study of Laplace's equation. At the same time, it is a ubiquitous equation, appearing in such diverse branches of physics as gravitation and magnetism, the theory of heat, and the study of soap bubbles.

3.1.1 Boundary Conditions and Uniqueness Theorems

Laplace's equation does not by itself determine V; in addition, a suitable set of boundary conditions must be supplied. The proof that a proposedset of boundary conditions will suffice is usually presented in the form of a **uniqueness theorem**. There are many such theorems for electrostatics, all sharing the same basic format-I'll show you the two most useful ones

First uniqueness theorem: The solution to Laplace's equation in some volume $\mathcal{V}$ is uniquely determined if V is specified on the boundary surface $\mathcal{S}$.

Proof: Imagine a region and its boundary, there could be "islands" inside, so long as V is given on all their surfaces; also the outer boundary could be at infinity, where V is ordinarily taken to be zero. Suppose there were two solutions to Laplace's equation

$$\nabla^2 V_1 = 0 \quad \nabla^2 V_2 = 0$$

both of which assume the specified value on the surface. I want to prove that they must be equal. The trick is to look at the difference

$$V_3 = V_1 - V_2$$

This obey's Laplace's equation

$$\nabla^2 V_3 = \nabla^2 V_1 - \nabla^2 V_2$$

and it takes the value zero on all the boundaries (since V_1 and V_2 are equal there). But Laplace's equation allows no maxima or minima-all extrema occur on the boundaries. So the maximum and minimum of V_3 are both zero. Therefore V_3 must be zero, and hence

$$V_1 = V_2$$

The uniqueness theorem is a license to your imagination. It doesnt matter how you come to your solution; if a) it satisfies Laplace's equation and (b) it has the correct values on the boundaries, then it's right.

3.1.2 Conductors and the Second Uniqueness Theorem

The simplest way to set boundaries for an electrostatic problem is to specify the value of V on all the surfaces surrounding the region of interest. And this situation often occurs in practice. In the laboratory, we have conductors connected to batteries, which maintain a given potebtial, or **ground**m which is the experimentalist's word for $V = 0$.

Second uniqueness theorem: In a volume $\mathcal{V}$ surrounded by conductors and containing a specified charge density ρ, the electric field is uniquely determined if the total charge on each conductor is given. (The region as a whole can be bounded by another conductor, or else unbounded.)

3.2 Multipole Expansion

3.2.1 Approximate Potentials at Large Distances

If you are very far away from a localized charge distribution, it "looks" like a point charge, and the potential is, to good approximation, $(1/4\pi\varepsilon_0)Q/r$, where Q is the total charge. We can expand the potential as

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{(n+1)}}\int (r')^n P_n(\cos\theta')\rho(\mathbf{r}')d\tau' \tag{3.2}$$

or more explicitely

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{1}{r}\int \rho(\mathbf{r}')d\tau' + \frac{1}{r^2}\int r'\cos\theta'\rho(\mathbf{r}')d\tau' + \frac{1}{r^3}\int (r')^2\left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right)\rho(\mathbf{r}')d\tau' + \ldots\right]$$

this is the desired result- the multipole expansion of V in powers of $1/r$. The first term $n = 0$ is the monopole contribution. The second $(n = 1)$ is the dipole term; the third is the quadrupole term; the fourth is the octopole term and so on. As it stands Equation 3.2 is exact, but it is useful primarily as an approximation scheme: The lowest nonzero term in the expansion provides the approximate potential at large r, and succesive terms tell us how to improve the approximation if greater precision is required.

3.2.2 The Monopole and Dipole Terms

3.3 Problems and Solutions

Problem # 1

Consider a cube made of 6 conducting plates of size $a \times a$, that encloses the space $-a/2 \leq x,y,z \leq a/2$. The plates at $z = \pm\, a/2$ are held at potentials $\pm V_0$, respectively, while the other four sides are all grounded $(V = 0)$.

(a). Solve the boundary value problem for $V(x,y,z)$ inside the cube.

First we must solve the Laplace equation

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

As always, we look for solutions that are products

$$V(x,y,z) = X(x)Y(y)Z(z)$$

Plugging this into the Laplace equation we find

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0$$

thus we know that

$$\frac{1}{X}\frac{d^2X}{dx^2} = c_1 \quad \frac{1}{Y}\frac{d^2Y}{dy^2} = c_2 \quad \frac{1}{Z}\frac{d^2Z}{dz^2} = c_3$$

which implies that

$$c_1 + c_2 + c_3 = 0$$

we can now solve this second order differential equation. One of these coefficients must be negative, this can only be c_3 because we know that this must be an oscillatory solution. In this case we find

$$c_3 = -(c_1 + c_2) \quad c_1 = l^2 \;\; c_2 = k^2 \quad c_3 = -(l^2 + k^2)$$

and so

$$\frac{d^2X}{dx^2} = l^2 X \quad \frac{d^2Y}{dy^2} = k^2 Y \quad \frac{d^2Z}{dz^2} = -(l^2 + k^2)Z$$

the solutions to these differential equations are

$$\begin{aligned} X(x) &= A\sin(lx) + B\cos(lx) \\ Y(y) &= C\sin(ky) + D\sin(ky) \\ Z(z) &= Ee^{\sqrt{l^2+k^2}z} + Fe^{-\sqrt{l^2+k^2}z} \end{aligned}$$

Boundary conditions

our boundary conditions are given as

$$\begin{aligned} \text{(i)}\ V &= 0 \ \text{ when } \ y = \pm a/2 \\ \text{(ii)}\ V &= 0 \ \text{ when } \ x = \pm a/2 \\ \text{(iii)}\ V &= \pm V_0 \ \text{ when } \ z = \pm a/2 \end{aligned}$$

from (i) we can see that

$$Y(a/2) = C\sin\left(l\frac{a}{2}\right) + D\cos\left(l\frac{a}{2}\right) = 0 \quad Y(-a/2) = -C\sin\left(l\frac{a}{2}\right) + D\cos\left(l\frac{a}{2}\right) = 0$$

and from (ii) we can see that

$$X(a/2) = A\sin\left(k\frac{a}{2}\right) + B\cos\left(k\frac{a}{2}\right) = 0 \quad X(-a/2) = -A\sin\left(k\frac{a}{2}\right) + B\cos\left(k\frac{a}{2}\right) = 0$$

from the first and second boundary condition we find

$$C = 0 \quad A = 0$$

and also

$$l = \frac{n\pi}{a} \qquad k = \frac{m\pi}{a} \qquad l^2 = \frac{n^2\pi^2}{a^2} \qquad k^2 = \frac{m^2\pi^2}{a^2}$$

where n and m are positive integers. Since the given boundary conditions imply an odd potential we can see that

$$Z(-z) = -Z(z) = \left[E^{-\sqrt{l^2+k^2}z} + Fe^{\sqrt{l^2+k^2}z}\right] = -\left[Ee^{\sqrt{l^2+k^2}z} + Fe^{-\sqrt{l^2+k^2}z}\right]$$

and so we find that

$$F = -E$$

and this now gives

$$Z(z) = E\left[e^{\sqrt{l^2+k^2}z} - e^{-\sqrt{l^2+k^2}z}\right] = E2\sinh(\sqrt{l^2+k^2}z)$$

combining the remaining constants we are left with

$$V(x,y,z) = C\cos\left(\frac{n\pi}{a}x\right)\cos\left(\frac{m\pi}{a}y\right)\sinh(\sqrt{l^2+k^2}z)$$

where C has absorbed the other constants, we can also write this as

$$V(x,y,z) = C\cos\left(\frac{n\pi}{a}x\right)\cos\left(\frac{m\pi}{a}y\right)\sinh\left(\frac{\pi}{a}\sqrt{n^2+m^2}z\right)$$

we can now write this as a double sum over the integers n and m

$$V(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{n,m} \sinh\left(\frac{\pi}{a}\sqrt{n^2+m^2}z\right)\cos(n\pi x/a)\cos(m\pi y/a)$$

and finally the last boundary conditions (iii) gives us

$$V(x,y,\pm a/2) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{n,m} \sinh\left(\frac{\pi}{2}\sqrt{n^2+m^2}\right)\cos(n\pi x/a)\cos(m\pi y/a) = \pm V_0 \qquad (3.3)$$

to determine the constant $C_{n,m}$ we multiply both sides by $\cos(n'\pi x/a)\cos(m'\pi y/a)$ where n' and m' are positive integers and integrate, this is known as Fouriers trick. For the left hand side we find

$$\sum_n\sum_m \sinh\left(\frac{\pi}{2}\sqrt{n^2+m^2}\right)C_{n,m}\int_{-a/2}^{a/2}\cos\left(\frac{n\pi}{a}x\right)\cos\left(\frac{n'\pi}{a}x\right)dx\int_{-a/2}^{a/2}\cos\left(\frac{m\pi}{a}y\right)\cos\left(\frac{m'\pi}{a}y\right)dy$$

we also know from Fouriers trick and orthogonality that $n = n' \;\; m = m'$ and so we get

$$\sinh\left(\frac{\pi}{2}\sqrt{n^2+m^2}\right)C_{n,m}\int_{-a/2}^{a/2}\cos^2\left(\frac{n\pi}{a}x\right)dx\int_{-a/2}^{a/2}\cos^2\left(\frac{m\pi}{a}y\right)dy = C_{n,m}\sinh\left(\frac{\pi}{2}\sqrt{n^2+m^2}\right)\frac{a^2}{4}$$

and now for the right hand side of equation 1 we find

$$\pm V_0\int_{-a/2}^{a/2}\cos\left(\frac{n'\pi}{a}x\right)dx\int_{-a/2}^{a/2}\cos\left(\frac{m'\pi}{a}y\right)dy = \pm V_0\frac{4a^2}{nm\pi^2}\sin\left(\frac{n\pi}{a}\right)\sin\left(\frac{m\pi}{a}\right)$$

and finally we can put these two solutions together to get

$$C_{n,m}\sinh\left(\frac{\pi}{2}\sqrt{n^2+m^2}\right)\frac{a^2}{4} = \pm V_0\frac{4a^2}{nm\pi^2}\sin\left(\frac{n\pi}{a}\right)\sin\left(\frac{m\pi}{a}\right)$$

this yields

$$C_{n,m} \;=\; \pm\frac{16V_0}{nm\pi^2}\frac{1}{\sinh\left(\frac{\pi}{2}\sqrt{n^2+m^2}\right)} \quad \text{for } n \text{ and } m \text{ odd}$$

if n and m are even then $C_{n,m}$ is zero. We can now write the potential as

$$V(x,y,z) = \boxed{\pm\frac{16V_0}{\pi^2}\sum_{n,m=1,3,5}\frac{1}{nm}\cos\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi}{a}y\right)\frac{\sinh\left(\frac{\pi}{a}\sqrt{n^2+m^2}z\right)}{\sinh\left(\frac{\pi}{2}\sqrt{n^2+m^2}\right)}}$$

(b). The potential is obviously zero at the center of the cube. What is the electric field there?

To find the electric field we use

$$\vec{E} = -\nabla V = \frac{\partial V}{\partial x}+\frac{\partial V}{\partial y}+\frac{\partial V}{\partial z}$$

and since we are evaluating the electric field at the origin we know that

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial y} = 0$$

this leaves us with

$$\vec{E}(x,y,z) = \frac{\partial V}{\partial z} = \pm\frac{16aV_0}{\pi}\sum_{n,m=1,3,5}\frac{1}{nm}\cos\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi}{a}y\right)\frac{\cosh\left(\frac{\pi}{a}\sqrt{n^2+m^2}z\right)}{\sinh\left(\frac{\pi}{2}\sqrt{n^2+m^2}\right)}\sqrt{n^2+m^2}\hat{\mathbf{z}}$$

and so the electric field at the origin is given by

$$\vec{E}(0,0,0) = \boxed{\pm\frac{16V_0}{a\pi}\sum_{n,m=1,3,5}\frac{1}{nm}\frac{\sqrt{n^2+m^2}}{\sinh\left(\frac{\pi}{2}\sqrt{n^2+m^2}\right)}\hat{\mathbf{z}}}$$

Problem # 2

(a). Suppose that on the surface of a sphere of radius R there is a surface charge density $\sigma(\theta) = \sigma_0\cos^2\theta$, where σ_0 is a constant. What is $V(r,\theta)$ inside and outside the sphere? What is the electric field $\mathbf{E}(r,\theta)$ inside and outside the sphere? Confirm that the field is discontinuous at the surface of the sphere in accord with the given charge distribution.

From equation 3.65 in Griffiths we know that

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta) \tag{3.4}$$

this is the general form of a potential in which we assumed **azimuthal symmetry** in spherical coordinates, this was derived in Griffiths on page 137-139. Looking at the inside of the sphere when $r < R$ we find that $B_l = 0$ for all l otherwise the potential will diverge. Looking at the potential outside the sphere we can see that $A_l = 0$ otherwise the potential would not satisfy the boundery condition that $V \to 0 \;\; r \to \infty$. From this we can see that

$$\begin{aligned} V(r &< R,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \\ V(r &\geq R,\theta) = \sum_{l=0}^{\infty}\frac{B_l}{r^{l+1}}P_l(\cos\theta) \end{aligned}$$

we know that the potentials are the same at $r = R$ and continous so we can set these expression equal to each other

$$\sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = \sum_{l=0}^{\infty}\frac{B_l}{R^{l+1}}P_l(\cos\theta)$$

and we can see that

$$B_l = A_l R^{2l+1}$$

since there is a discontinuity in the radial derivative we can use equation 2.36

$$\left(\frac{\partial V_{out}}{\partial r} - \frac{\partial V_{in}}{\partial r}\right)_{r=R} = -\frac{1}{\varepsilon_0}\sigma_0(\theta)$$

from this we find

$$-\sum_{l=0}^{\infty}(l+1)\frac{B_l}{R^{l+2}}P_l(\cos\theta)-\sum_{l=0}^{\infty}lA_lR^{l-1}P_l(\cos\theta)=-\frac{1}{\varepsilon_0}\sigma_0\theta$$

substituting the above expression for B_l and some algebra yields

$$\sum_{l=0}^{\infty}(2l+1)A_lR^{l-1}P_l(\cos\theta)=\frac{1}{\varepsilon_0}\sigma_0\cos^2\theta$$

we can use our knowledge of Legendre polynomials to solve this problem

$$\cos^2\theta=\left(P_2(\cos\theta)+\frac{1}{2}\right)\frac{2}{3}=\left(P_2(\cos\theta)+\frac{P_0(\cos\theta)}{2}\right)\frac{2}{3}$$

looking at only the $l=0,2$ terms in the sum and using this on the above expression yields

$$A_0R^{-1}P_0(\cos\theta)+5A_2RP_2(\cos\theta)=\frac{1}{3}\frac{\sigma_0}{\varepsilon_0}P_0(\cos\theta)+\frac{2}{3}\frac{\sigma_0}{\varepsilon_0}P_2(\cos\theta)$$

this allows us to solve for A_0 and A_2 as

$$A_0=\frac{\sigma_0R}{3\varepsilon_0}\quad A_2=\frac{2\sigma_0}{15\varepsilon_0}R^{-1}$$

so the potentials for inside and outside the spheres are given as

$$\begin{aligned}V(r &< R,\theta)=\frac{\sigma_0R}{3\varepsilon_0}+\frac{2\sigma_0r^2}{15\varepsilon_0R}P_2(\cos\theta)\\ V(r &\geq R,\theta)=\frac{\sigma_0R^2}{3\varepsilon_0r}+\frac{2\sigma_0R^4}{15\varepsilon_0r^3}P_2(\cos\theta)\end{aligned}$$

We can find the electric field by

$$\vec{E}=-\nabla V=-\left[\frac{\partial V}{\partial r}\hat{\mathbf{r}}+\frac{1}{r}\frac{\partial V}{\partial\theta}\hat{\theta}\right]$$

for inside the sphere we find

$$\vec{E}_{in}=-\nabla V_{in}=-\frac{4}{15}\frac{r\sigma_0}{R\varepsilon_0}\left(\frac{3\cos^2\theta-1}{2}\right)\hat{\mathbf{r}}+\frac{r\sigma_0 3\sin(2\theta)}{5R\varepsilon_0}\hat{\theta}$$

knowing that the derivative of the θ term is

$$\frac{d}{d\theta}\frac{3}{2}\cos^2\theta=\frac{3}{2}\frac{d}{d\theta}\frac{1}{2}[1+\cos2\theta]=\frac{3}{2}\sin2\theta$$

which can be simplified as

$$\vec{E}_{in}=-\nabla V_{in}=\boxed{\frac{2}{15}\frac{r\sigma_0}{R\varepsilon_0}\left[\frac{3}{2}\sin(2\theta)\hat{\theta}-[3\cos^2\theta-1]\hat{\mathbf{r}}\right]}$$

for the outside we find

$$\vec{E}_{out} = -\nabla V_{out} = \boxed{-\left[\left(-\frac{\sigma_0}{3\varepsilon_0}\frac{R^2}{r^2} - \frac{1}{5}\frac{R^4\sigma_0}{r^4\varepsilon_0}(3\cos^2\theta - 1)\right)\hat{\mathbf{r}} + \frac{1}{5}\frac{R^4\sigma_0}{r^4\varepsilon_0}(\sin 2\theta)\hat{\theta}\right]}$$

to show that these fields are discontinous at the surface, one would just have to show that at $r = R$ $\vec{E}_{in} \neq \vec{E}_{out}$ and this can be seen from the above expressions.

(b). Now repeat the above exercise for the surface charge density $\sigma(\theta) = \sigma_0 \sin^2\theta$. Note that in this case

$$V(r,\theta) \;\; = \left(\begin{array}{l} \frac{2R\sigma_0}{3\varepsilon_0} - \frac{2R\sigma_0}{15\varepsilon_0}\left(\frac{r}{R}\right)^2 P_2(\cos\theta) \;\; r < R \\ \frac{2R\sigma_0}{3\varepsilon_0} - \frac{2R\sigma_0}{15\varepsilon_0}\left(\frac{R}{r}\right)^3 P_2(\cos\theta) \;\; r \geq R \end{array}\right)$$

For this problem we can just start with

$$\sum_{l=0}^{\infty}(2l+1)A_l R^{l-1} P_l(\cos\theta) = \frac{\sigma_0 \sin^2\theta}{\varepsilon_0} = \frac{\sigma_0}{\varepsilon_0}(1-\cos^2\theta)$$

and only looking at the $l = 0,2$ case in the sum we find

$$A_0 R^{-1} P_o(\cos\theta) + 5A_2 R P_2(\cos\theta) = \frac{\sigma_0}{\varepsilon_0}\left(P_0(\cos\theta) - \frac{2}{3}P_2(\cos\theta) - \frac{1}{3}P_0(\cos\theta)\right)$$

setting like terms equal to each other gives

$$A_0 = \frac{2\sigma_0 R}{3\varepsilon_0} \quad A_2 = -\frac{2\sigma_0}{15R\varepsilon_0}$$

and so finally we find that the potentials are given by

$$V(r < R,\theta) = \boxed{\frac{2R\sigma_0}{3\varepsilon_0} - \frac{2R\sigma_0}{15\varepsilon_0}\left(\frac{r}{R}\right)^2 P_2(\cos\theta)}$$

and also

$$V(r \geq R,\theta) = \boxed{\frac{2R^2\sigma_0}{3\varepsilon_0 r} - \frac{2R\sigma_0}{15\varepsilon_0}\left(\frac{R}{r}\right)^3 P_2(\cos\theta)}$$

and the electric field is given as

$$\vec{E}_{in} = -\nabla V_{in} = \boxed{-\frac{2r\sigma_0 R}{15\varepsilon_0}(3\cos^2\theta - 1)\hat{\mathbf{r}} + \frac{1}{5}\frac{\sigma_0 r^2}{\varepsilon_0}(\sin 2\theta)\hat{\theta}}$$

and for the outside we have

$$\vec{E}_{out} = -\nabla V_{out} = \boxed{-\left[\left(-\frac{2R^2\sigma_0}{3\varepsilon_0 r^2} + \frac{6\sigma_0 R^4}{15\varepsilon_0 r^4}(\cos^2\theta - 1)\right)\hat{\mathbf{r}} + \frac{1}{5}\frac{\sigma_0 R^4}{\varepsilon_0 r^4}\sin 2\theta\hat{\theta}\right]}$$

Problem # 3

Solutions to Laplace's equation in two dimensions using complex functions: One sometimes encounters a boundary-value problem that is essentially two-dimensional, in that, by the nature of the problem, we can assume the electrostatic potential is constant along one Cartesian direction. In this case, it may be possible to find the solution to our boundary-value problem among the many solutions obtained by analysis of analytic functions of a complex variable. For this, let's consider the complex number $\zeta = x + iy$, where x and y are the remaining two Cartesian coordinates for which we wish to determine $V(x,y)$. We let $F(\zeta)$ be an analytic function of ζ- meaning that it is a well-behaved function with well defined derivatives.

(a). Show that $F(\zeta)$ satisfies Laplace's equation, i.e. that

$$\nabla^2 F = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = 0$$

To show this we begin with

$$F(\xi) = F(x+iy) \quad \xi = x+iy$$

we also know that

$$\begin{aligned} \frac{\partial F}{\partial x} &= \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = \frac{\partial F}{\partial z} \cdot 1 \\ \frac{\partial^2 F}{\partial x^2} &= \frac{\partial}{\partial x}\left[\frac{\partial F}{\partial z}\right] = \frac{\partial}{\partial z}\left[\frac{\partial F}{\partial x}\right] = \frac{\partial^2 F}{\partial z^2} \end{aligned}$$

and now for y we find

$$\begin{aligned} \frac{\partial F}{\partial y} &= \frac{\partial F}{\partial z}\frac{\partial z}{\partial y} = \frac{\partial F}{\partial z} \cdot i \\ \frac{\partial^2 F}{\partial y^2} &= i \cdot \frac{\partial}{\partial y}\left[\frac{\partial F}{\partial z}\right] = \frac{\partial}{\partial z}\left[\frac{\partial F}{\partial y}\right] = i^2 \cdot \frac{\partial^2 F}{\partial z^2} = -\frac{\partial^2 F}{\partial z^2} \end{aligned}$$

because we know that the differential operator is commutative and we can say that

$$\frac{\partial}{\partial x}\frac{\partial}{\partial y} = \frac{\partial}{\partial y}\frac{\partial}{\partial x}$$

so we have shown that *all* nice well behaved complex functions satisfies Laplaces equation.

$$\nabla^2 F = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = \frac{\partial^2 F}{\partial z^2} - \frac{\partial^2 F}{\partial z^2} = 0$$

Given this fact, we have indeed found two solutions to Laplace's equation in two dimensions. Expressing $F = g(x,y) + ih(x,y)$, then the real-valued functions g and h are both such solutions.

Deriving from a single complex analytic function, the functions $g(x,y)$ and $h(x,y)$, known as conjugate functions, possess an important relation to one another. By considering first derivatives of $F(\zeta)$ one can demonstrate the validity of the Cauchy-Riemann relations:

$$\frac{\partial g}{\partial x} = \frac{\partial h}{\partial y} \qquad \frac{\partial g}{\partial y} = -\frac{\partial h}{\partial x}$$

(b). Show that the electric field obtained by setting $V(x,y) = g(x,y)$ is everywhere orthogonal to that obtained for $V(x,y) = h(x,y)$. Based on this observation, one can state that the curves $h(x,y) = h_0$ describe the field lines for the potential $V(x,y) = g(x,y)$, and visa versa.

To show that these two solutions are orthogonal we must take the dot products of the two solutions

$$\begin{aligned}\vec{E}(g) &= -\nabla V = -\nabla g(x,y) = -\left[\frac{\partial g}{\partial x}\hat{\mathbf{x}} + \frac{\partial g}{\partial y}\hat{\mathbf{y}}\right] \\ \vec{E}(h) &= -\nabla V = -\nabla h(x,y) = -\left[\frac{\partial h}{\partial x}\hat{\mathbf{x}} + \frac{\partial h}{\partial y}\hat{\mathbf{y}}\right]\end{aligned}$$

the dot product must be zero for these two solutions to be orthogonal.

$$\vec{E}(g)\cdot\vec{E}(h) = \left[\frac{\partial g}{\partial x}\hat{\mathbf{x}} + \frac{\partial g}{\partial y}\hat{\mathbf{y}}\right]\cdot\left[\frac{\partial h}{\partial x}\hat{\mathbf{x}} + \frac{\partial h}{\partial y}\hat{\mathbf{y}}\right] = \frac{\partial g}{\partial x}\frac{\partial h}{\partial x} + \frac{\partial g}{\partial y}\frac{\partial h}{\partial y} = 0$$

but since we know that the Cauchy-Reimann equations are

$$\frac{\partial g}{\partial x} = \frac{\partial h}{\partial y} \quad \frac{\partial h}{\partial x} = -\frac{\partial g}{\partial y}$$

using this we find

$$\frac{\partial g}{\partial x}\frac{\partial h}{\partial x} - \frac{\partial g}{\partial x}\frac{\partial h}{\partial x} = 0$$

(c). One application of this method: Find the potential $V(x,y)$ in the wedge-shaped region between two plates (infinite half-planes) that intersect at the z axis at an angle$\alpha < \pi/2$, and are both at the potential V_0 . You might consider the real or imaginary parts of the complex function $F(\zeta) = A\zeta^p + B$ and attempt to match the given boundary conditions. Other nice solutions obtained by this method are the field produced at the edge of a semi- infinite charged conducting plate, and the fringing field of a parallel plate capacitor.

Since we know that we can write a complex function as

$$F(\xi) = F(x+iy) = Ar^p e^{ip\phi} + B \qquad \xi = re^{i\phi} = r[\cos(\phi) + i\sin(\phi)]$$

where ϕ is now the angle defined in the complex plane and r is the distance from the origin, thus we find

$$F(\xi) = Ar^p[\cos(p\phi) + i\sin(p\phi)] + B = Ar^p\cos(p\phi) + B + Ar^p i\sin(p\phi)$$

since we know that both the real solution and imaginary solution satisfy the Laplace equation we can solve for the real part and we are guranteed that this is a unique solution.

$$\mathbb{R}\, V(r,\phi) = Ar^p\cos(p\phi) + B \tag{3.5}$$

Imposing boundary conditions we find

$$\begin{aligned}V(r,\phi &= 0) = Ar^p + B = V_0 \\ V(r,\phi = \alpha) &= Ar^p\cos(p\alpha) + B = V_0\end{aligned}$$

Subtracting the first equation from the second we find

$$Ar^p - Ar^p\cos(p\alpha) = 0 \quad p = \frac{2n\pi}{\alpha}$$

now if we add the two solutions we find

$$Ar^p + Ar^p\cos(p\alpha) + 2B = 2V_0$$

using the expression for p we get

$$2Ar^p + 2B = 2V_0 \qquad B = V_0 - Ar^p$$

inserting this into equation 3 we find

$$Ar^p\cos(p\phi) + V_0 - Ar^p = Ar^p(\cos(p\phi) - 1) + V_0$$

we can also see that

$$x = r\cos(p\phi) \qquad \cos(p\phi) = \frac{x}{r} \qquad r = \sqrt{x^2+y^2}$$

and so we find

$$V(x,y) = \boxed{Ax(x^2+y^2)^{(p-1)/2} - A(x^2+y^2)^{p/2} + V_0}$$

Problem # 4

Solve Laplace's equation by separation of variables in *cylindrical* coordinates, assuming there is no dependence on z (cylindrical symmetry). [Make sure you find *all* solutions to the radial equation; in particular, your result must accomidate the case of an infinite line charge, for which of course we already know the answer.]

We can begin by writing the potential as a product

$$V(s,\phi) = S(s)\Phi(\phi)$$

and solving the Laplace equation in cylindrical coordinates

$$\nabla^2 V = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial V}{\partial\phi^2} = 0$$

substituting our potential into the Laplace equation and multiplying every term by s^2 yields

$$\frac{s}{S(s)}\frac{d}{ds}\left(s\frac{dS(s)}{ds}\right) + \frac{1}{\Phi(\phi)}\frac{d\Phi(\phi)}{d\phi^2} = \frac{s}{S(s)}\left(s\frac{d^2S(s)}{ds} + \frac{dS(s)}{ds}\right) + \frac{1}{\Phi(\phi)}\frac{d\Phi(\phi)}{d\phi^2} = 0$$

and letting

$$\frac{s}{S(s)}\left(s\frac{d^2S(s)}{ds} + \frac{dS(s)}{ds}\right) = p_1 \qquad \frac{1}{\Phi(\phi)}\frac{d\Phi(\phi)}{d\phi^2} = p_2 \qquad p_1 + p_2 = 0$$

if we now let $p_1 = \lambda^2$ and $p_2 = -\lambda^2$,we chose these constants because we were motivated by knowing the angular solution should be oscillatory. We can now write

$$s^2\frac{d^2S(s)}{ds} + s\frac{dS(s)}{ds} = \lambda^2 S(s) \qquad \frac{d\Phi(\phi)}{d\phi^2} = -\lambda^2\Phi(\phi)$$

we can see that the angular solutions are given by

$$\Phi(\phi) = A'e^{i\lambda\phi} + B'e^{-i\lambda\phi} = A\sin(\lambda\phi) + B\cos(\lambda\phi) \quad \lambda \geq 1$$

where $A \neq A'$ and $B \neq B'$. Now to solve the radial differential equation, which is a non-linear homogeneus second order differential equation. We can write it in a more suggestive form

$$s^2 S''(s) + sS'(s) - \lambda^2 S(s) = 0$$

and now we can assume that it has a polynomial form

$$S(s) = s^m \quad S'(s) = ms^{m-1} \quad S''(s) = m(m-1)s^{m-2}$$

substituting this into our equation gives

$$\begin{array}{rcl} s^2 m(m-1)s^{m-2} + sms^{m-1} - \lambda^2 s^m & = & 0 \\ m(m-1) + m - \lambda^2 & = & 0 \\ m & = & \pm\lambda \end{array}$$

so we find that the radial solution is given by

$$S(s) = Cs^{\lambda} + Ds^{-\lambda} \quad \lambda \geq 1$$

we have only found the solutions where $\lambda \geq 1$, what about the solutions where $\lambda = 0$? We can solve this by solving the same differential equations for this special case.

$$\frac{d^2\Phi(\phi)}{d\phi^2} = 0 \quad \frac{d}{ds}\left(s\frac{dS(s)}{ds}\right) = 0$$

for these solutions to hold we must set the first derivative equal to a constant

$$\frac{d\Phi(\phi)}{d\phi} = c \quad \Phi(\phi) = c\phi + k \quad \lambda = 0$$

but since we know

$$\Phi(\phi) = \Phi(\phi + 2\pi)$$

tells us that $c = 0$, which gives

$$\Phi(\phi) = k$$

where k is just a constant. Now for the radial part

$$s\frac{dS(s)}{ds} = d \quad S(s) = d\ln s + j \quad \lambda = 0$$

we have now found *all* solutions to the potential which can now be written as

$$V(s,\phi) = \boxed{k + d\ln s + \sum_{\lambda=1}^{\infty} [C_\lambda s^{\lambda} + D_\lambda s^{-\lambda}][A_\lambda \sin(\lambda\phi) + B_\lambda \cos(\lambda\phi)]}$$

To show that our solution accomidates an infinite line charge. We know that the electric field of an infinite line charge is given by

$$\vec{E} = \frac{\lambda}{2\pi s\varepsilon_0}\hat{\mathbf{s}}$$

and the potential is

$$V = -\int_0^s \vec{E}\cdot d\mathbf{s} = \frac{\lambda}{2\pi\varepsilon_0}\ln s + k$$

and since we know that by symmetry, and also the fact that a line does not any well defined angles we get

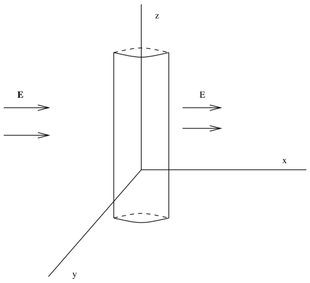
$$\sum_{\lambda=1}^{\infty}[C_\lambda s^\lambda + D_\lambda s^{-\lambda}][A_\lambda \sin(\lambda\phi) + B_\lambda \cos(\lambda\phi)] = 0$$

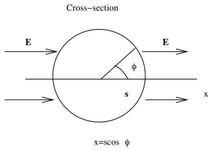
we find

$$V(s,\phi) = k + d\ln s$$

Problem # 5

Find the potential outside an infinitely long metal pipe, of radius R, placed at right angles to an otherwise uniform electric field $\mathbf{E_0}$. Find the surface charge induced on the pipe.





Given the solution found from the last problem

$$V(s,\phi) = k + d\ln s + \sum_{\lambda=1}^{\infty}[C_\lambda s^\lambda + D_\lambda s^{-\lambda}][A_\lambda \sin(\lambda\phi) + B_\lambda \cos(\lambda\phi)]$$

the surface of the cylinder is equipotential and thus we can just set the potential equal to zero. We also know that the potential does not go to zero at large x, far from the cylinder the potential is

$$V \to -E_0 x + C \qquad x = s\cos\theta$$

We need to apply the boundary conditions

$$\begin{aligned} \text{(i) } V &= 0 \quad \text{when } s = R \\ \text{(ii) } V &= -E_0 s\cos\phi \quad \text{when } s \gg R \end{aligned}$$

From boundary condition (i) we find

$$V(R,\phi) = k + d\ln R + \sum_{\lambda=1}^{\infty}[C_\lambda R^\lambda + D_\lambda R^{-\lambda}][A_\lambda \sin(\lambda\phi) + B_\lambda \cos(\lambda\phi)] = 0$$

and from condition (ii) we find

$$V(s >> R,\phi) = k + d\ln s + \sum_{\lambda=1}^{\infty}[C_\lambda s^\lambda + D_\lambda s^{-\lambda}][A_\lambda \sin(\lambda\phi) + B_\lambda \cos(\lambda\phi)] = -E_0 s\cos\theta$$

where the term $D_\lambda s^{-\lambda}$ becomes negligable for $s \gg R$, from conditions (i) and (ii) we can see that

$$k = 0 \quad d = 0 \quad A_\lambda = 0$$

and also that $\lambda = 1$. This gives us

$$V(s,\phi) = Cs + Ds^{-1})B\cos(\phi)$$

we can now absorb B into C and D we get

$$V(s,\phi) = (C's + D's^{-1})\cos(\phi)$$

applying the boundary conditions again gives us

$$\begin{array}{rcl} \text{(i)} \quad C'R + \dfrac{D'}{R} & = & 0 \quad D' = -C'R^2 \\ \text{(ii)} \quad C's\cos(\phi) & = & -E_0 s\cos\phi \quad C' = -E_0 \end{array}$$

this works because we can ignore the D'/s term because it is negligable.

$$V(s,\phi) = \left(-E_0 s + \frac{E_0 R^2}{s}\right)\cos(\phi)$$

To find the induced charge we can use

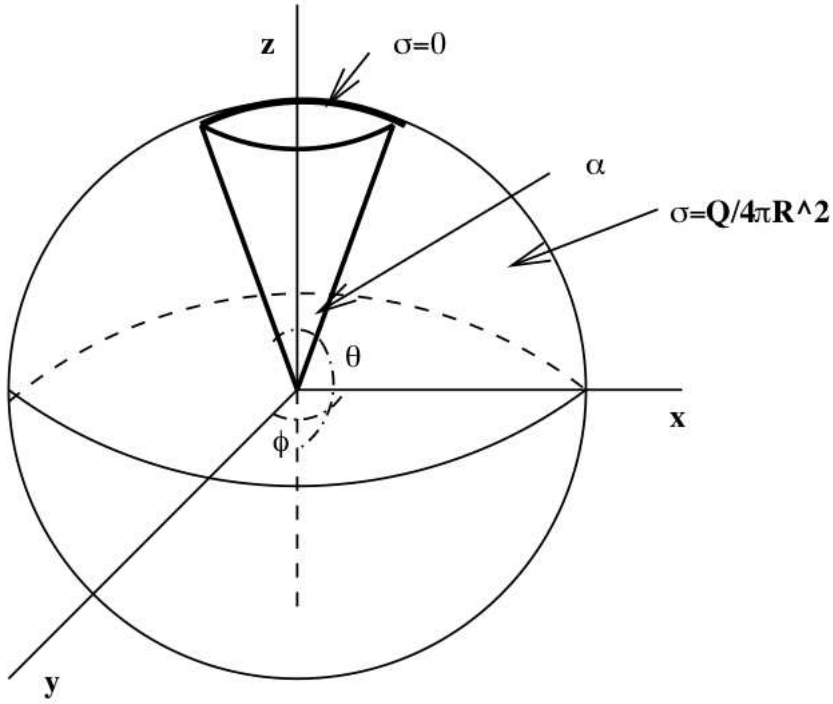
$$\sigma = -\varepsilon_0 \frac{\partial V}{\partial s}$$

we know tha

$$\begin{array}{rcl} \dfrac{\partial V}{\partial s} & = & \cos\phi\left[-E_0 - E_0\dfrac{R^2}{s^2}\right] \\ \sigma & = & -\varepsilon_0\cos\phi\left[-E_0 - E_0\dfrac{R^2}{s^2}\right] \quad s = R \\ \sigma & = & \boxed{2\varepsilon_0 E_0\cos\phi} \end{array}$$

Problem # 6

A spherical surface of radius R has charge uniformly distributed over its surface with a density $Q/4\pi R^2$, except for a spherical cap at the north pole, defined by the bounding cone $\theta = \alpha$.



(a). Before we get started here, show the following recursion relation for Legendre polynomials to be true:

$$\frac{dP_{l+1}(x)}{dx} - \frac{dP_{l-1}(x)}{dx} - (2l+1)P_l(x) = 0 \tag{3.6}$$

(for $l > 0$). This can be obatined by considering the Rodriguez formula. (Griffiths 3.62)

The Rodriguez formula (Equation 3.62 in Griffiths) is given as

$$P_l(x) = \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l (x^2-1)^l$$

taking the first derivative with respect to x on $P_{l+1}(x)$ yields

$$\frac{d}{dx}P_{l+1}(x) = \frac{1}{2^l 2(l+1)!}\left(\frac{d}{dx}\right)^l \frac{d^2}{dx^2}(x^2-1)^{l+1} = \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l [(x^2-1)^l + 2lx^2(x^2-1)^{l-1}]$$

in the above calculation we used the following relationships

$$(l+1)! = (l+1)l! \quad \frac{1}{(l-1)!} = \frac{l}{l!}$$

taking the first derivative with respect to x on $P_{l-1}(x)$ yields

$$\frac{d}{dx}P_{l-1}(x) = \frac{2}{2^l(l-1)!}\left(\frac{d}{dx}\right)^l (x^2-1)^{l-1} = \frac{2l}{2^l l!}\left(\frac{d}{dx}\right)^l (x^2-1)^{l-1}$$

and finally expanding the last term gives

$$(2l+1)P_l(x) = (2l+1)\frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l (x^2-1)^l = (2l+1)\frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l (x^2-1)^{l-1}(x^2-1)$$

putting all this together yields

$$\begin{aligned}
\left(\frac{d}{dx}\right)^l \left[(x^2-1)^l + 2lx^2(x^2-1)^{l-1} - (x^2-1)^{l-1} - [(2l+1)(x^2-1)^l\right] &= 0 \\
\left(\frac{d}{dx}\right)^l \left[2lx^2(x^2-1)^{l-1} - (x^2-1)^{l-1} - 2lx^2(x^2-1)^{l-1} + (x^2-1)^{l-1}\right] &= 0 \\
0 &= 0
\end{aligned}$$

(b). Show that the potential inside the spherical surface can be expressed as

$$V = \frac{Q}{2}\sum_{l-0}^{\infty}\frac{1}{2l+1}[P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)\frac{r^l}{R^{l+1}}P_l(\cos\theta)$$

where, for $l=0$, we should substitute $P_{l-1}(\cos\alpha) = -1$.

Integrating every term in Equation 1 gives

$$\int P_l(x)dx = \frac{1}{2l+1}\left[P_{l+1}(x) - P_{l-1}(x)\right]$$

Knowing that the surface charge density and the general expression of the potential inside a sphere are given by

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \qquad \sigma = \frac{Q}{4\pi R^2}$$

motivated by example 3.9 in Griffiths we know that the coefficients A_l are given by equation 3.84

$$A_l = \frac{1}{2\varepsilon R^{l-1}}\int_0^{\pi}\sigma(\theta)P_l(\cos\theta)\sin\theta d\theta$$

we also know from Fouriers trick

$$\int_a^b P_l\cos\theta\sin\theta d\theta = -\int_a^b P_l(x)dx = \int_b^a P_l(x)dx$$

We find

$$A_l = \frac{1}{2\varepsilon R^{l-1}}\int_0^{\alpha}\sigma(\theta)P_l(\cos\theta)\sin\theta d\theta + \frac{1}{2\varepsilon R^{l-1}}\int_{\alpha}^{\pi}\sigma(\theta)P_l(\cos\theta)\sin\theta d\theta$$

the first integral is just zero because the surface charge density is zero there and the second integral is given by

$$A_l = \frac{1}{2\varepsilon R^{l-1}}\int_{\alpha}^{\pi}\sigma(\theta)P_l(\cos\theta)\sin\theta d\theta = \frac{\sigma(\theta)}{2\varepsilon R^{l-1}}\int_{-1}^{\cos\alpha}P_l(x)dx$$

where the integration limits are given by the change of variable of the integral. From the relationship from part (a) we find

$$A_l = \frac{\sigma}{2\varepsilon(2l+1)}\left[P_{l+1}\cos\alpha - P_{l-1}\cos\alpha\right] = \frac{Q}{8\pi\varepsilon(2l+1)R^{l+1}}\left[P_{l+1}\cos\alpha - P_{l-1}\cos\alpha\right]$$

where $P_{l+1}(-1) - P_{l-1}(-1) = 0$, putting this into the general expression for the potential inside a sphere yields

$$V(r < R, \theta) = \boxed{\frac{Q}{8\pi\varepsilon} \sum_{l=0}^{\infty} \frac{r^l P_l(\cos\theta)}{(2l+1)R^{l+1}} [P_{l+1}\cos\alpha - P_{l-1}\cos\alpha]}$$

(c). What is the potential outside the sphere?

We know that the general expression for the potential outside of a sphere is given by

$$V(r > R, \theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

but we know that the coefficients B_l (Equation 3.81) can be expressed as

$$B_l = A_l R^{2l+1}$$

thus

$$V(r > R, \theta) = \sum_{l=0}^{\infty} \frac{A_l R^{2l+1}}{r^{l+1}} P_l(\cos\theta) = \boxed{\frac{Q}{8\pi\varepsilon} \sum_{l=0}^{\infty} \frac{R^l}{(2l+1)r^{l+1}} [P_{l+1}\cos\alpha - P_{l-1}\cos\alpha]}$$

(d). Find the magnitude and the direction of the electric field at the origin.

We know that the electric field is given by

$$\mathbf{E} = -\nabla V = -\frac{\partial V}{\partial r}$$

Since θ is not well defined at the origin and we know that the electric field points in the $\hat{\mathbf{z}}$ direction and we do not compute the derivative with respect to θ. The electric field is given by

$$\mathbf{E}_{in} = -\frac{Q}{8\pi\varepsilon} \sum_{l=1}^{\infty} \frac{l r^{l-1} P_l \cos\theta}{(2l+1)R^{l+1}} [P_{l+1}\cos\alpha - P_{l-1}\cos\alpha]\hat{\mathbf{r}}$$

and so we know $\hat{\mathbf{r}} = \hat{\mathbf{z}}$, and $\theta = 0$. We can also see that the above expression tells us the $l = 1$ in order for the electric field to not blow up at the origin or go to zero.

$$\begin{aligned}
\mathbf{E} &= -\frac{Q}{8\pi\varepsilon R^2}\frac{1}{3}[P_2\cos\alpha - P_0\cos\alpha]\hat{\mathbf{z}} \\
&= -\frac{Q}{8\pi\varepsilon R^2}\frac{1}{3}\left[\left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right) - 1\right]\hat{\mathbf{z}} \\
&= -\frac{Q}{4\pi\varepsilon R^2}\frac{1}{4}[\cos^2\alpha - 1]\hat{\mathbf{z}} \\
\mathbf{E} &= \boxed{\frac{Q}{16\pi\varepsilon R^2}[1 - \cos^2\alpha]\hat{\mathbf{z}}}
\end{aligned}$$

and the magnitude is given by

$$|E| = \boxed{\frac{Q}{16\pi\varepsilon R^2}[1 - \cos^2\alpha]}$$

(e). Discuss the limiting forms of the potential (b,c) and the electric field (d) as the spherical cap becomes either (1) very small or (2) so large that the entire remaining charged area becomes a very small cap at the south pole.

We can see from the figure that if α gets very small the sphere would be almost represented as a spherical conductor and so

$$\alpha \lll 1 \quad \alpha \text{ very small}$$

the potentials and the electric field would behave as

$$\begin{aligned} V(r &< R,\theta) \approx V_0 \\ V(r &> R,\theta) \propto \frac{1}{r} \\ E(r &< R,\theta) \approx 0 \end{aligned}$$

and if the cap becomes very large the sphere would be represented by a point charge at the south pole of the sphere. The potentials and electric field would behave as

$$\begin{aligned} V(r &< R,\theta) \propto \frac{1}{r} \\ V(r &> R,\theta) \propto \frac{1}{r} \\ E(r &< R,\theta) \propto \frac{1}{r^2} \end{aligned}$$

Problem # 7

Many nuclei have electric quadrupole moments. The quadrupole moment of a given state of a given nucleus is typically quantified as $Q = Q_{33}/e$ where Q_{ij} is defined as in Griffiths' Problem 3.45, and the index "3" refers to the $\hat{\mathbf{z}}$ axis. Now, consider a nucleus centered at the origin finding itself in a cylindrically symmetric electric field with a gradient $\partial E_z/\partial z$ along the $\hat{\mathbf{z}}$ axis at the position of the nucleus.

(a). Show that the inhomogeneous field modifies the energy of the quadrupole by

$$W = -\frac{e}{4}Q\left(\frac{\partial E_z}{\partial z}\right)$$

For this, you might want to find a suitable form of the electrostatic potential near the position of the nucleus. Be sure that this potential satisfies the Laplace equation. You should let $V = 0$ at the origin.

We know that energy of a continuous charge distribution can be written as

$$W = \int \rho(r')\Phi(r')d\tau'$$

where $\Phi(r')$ is the externel potential energy, where the general form can be expressed as

$$\Phi(r) = \frac{1}{4\pi\varepsilon}\sum_{l=0}^{\infty}\frac{1}{r^{n+1}}\int (r')^n P_n(\cos\theta')\rho(r')d\tau'$$

we can expend this using Taylor expansion to find

$$\Phi(0) = \Phi(0) + r'\cdot\nabla\Phi(0) + \frac{1}{2}\sum_{i,j} r_i r_j \partial_i \partial_j \Phi(0)$$

where we know that the last term is the quadrupole term, which is the one we are concerned about. Plugging this into the work function yields

$$W = \frac{1}{2}\int \rho(r')\sum_{i,j} r_i r_j \partial_i \partial_j \Phi(0) d\tau'$$

and since we are only concerned with Q_{33} we know that $i = j = 3$, thus we find

$$W = \frac{1}{2}\int \rho(r')d\tau' \left[(\vec{r}_3)^2 \frac{\partial^2}{\partial z^2}\Phi(0)\right]$$

but we know that

$$\frac{\partial^2}{\partial z^2}\Phi(0) = -\frac{\partial E_z}{\partial z}$$

putting this into the above expression yields

$$W = -\frac{1}{2}\frac{\partial E_z}{\partial z}\int (\vec{r}_3)^2 \rho(r')d\tau'$$

using the relationship from problem 3.45

$$Q_{33} = \int [3(\vec{r}_3)^2 - (\vec{r}_3)^2]\rho(r')d\tau' \quad \Rightarrow \quad \frac{Q_{33}}{2} = \int \rho(r')d\tau'(\vec{r}_3)^2$$

using this expression we find that the work is given by

$$W = -\frac{Q_{33}}{4}\frac{\partial E_z}{\partial z}$$

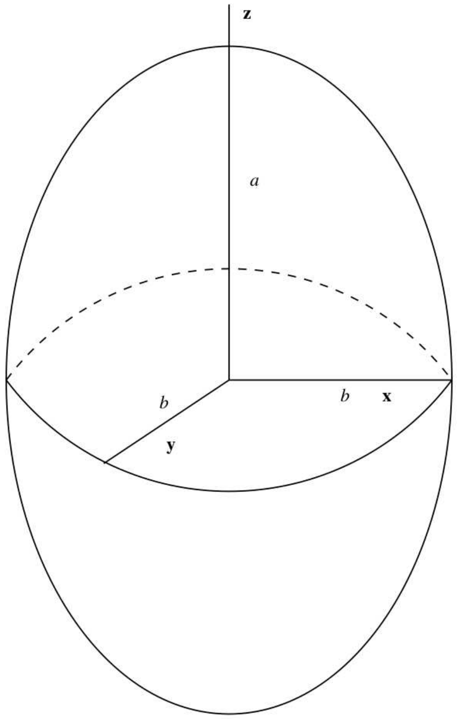
since we know that

$$Q_{33} = eQ$$

we find

$$W = \boxed{-\frac{eQ}{4}\frac{\partial E_z}{\partial z}}$$

(b) Nuclear charge distributions can be approximated by a constant charge density throughout a spheroidal (egg-shaped) volume of semimajor axis a and semimajor axis b. Calculate the quadrupole moment of such a nucleus assuming that the total charge is Ze. Given that the ^{153}Eu$(Z = 63)$ nucleus has a quadrupole moment of $Q = 2.5 \times 10^{-25}\text{cm}^2$ and a mean radius $R = \frac{a+b}{2} = 7 \times 10^{-13}\text{cm}$, determine the fractional difference in radius $(a-b)/R$.



Using the definition of the quadrupole moment

$$Q_{ij} = \int [3r_i' r_j' - (r')^2 \delta_{ij}] \rho(r') d\tau'$$

and since we are only looking for Q_{zz} we find

$$Q_{zz} = \rho(r) \int [2z^2 - x^2 - y^2] dxdydz$$

This gives us the quadrupole moment in cartesian coordinates, it would be easier to solve this integral in spherical coordinates, we can do a linear transformation to do this, the equation that models our spheroid is given by

$$\frac{x^2}{b^2} + \frac{y^2}{b^2} + \frac{z^2}{a^2} = 1$$

letting

$$u = \frac{x}{b} \quad v = \frac{y}{b} \quad w = \frac{z}{a}$$

where the derivatives are given by

$$dx = bdu \quad dy = bdv \quad dz = adw$$

and so we know

$$dxdydz = ab^2 dudvdw$$

we also know that the cartesian components in spherical coordiantes are given by

$$\begin{aligned} u &= r\sin\theta\cos\phi \\ v &= r\sin\theta\sin\phi \\ w &= r\cos\theta \end{aligned}$$

using the above relationships we get

$$Q_{zz} = \rho(r)\int\int\int [2(wa)^2-(bv)^2-(bu)^2]ab^2 d\tau$$

which gives

$$\rho(r)\int_0^{2\pi}\int_0^{\pi}\int_0^{1}[2a^2r^2\cos^2\theta - b^2r^2\sin^2\theta\cos^2\phi - b^2r^2\sin^2\theta\sin^2\phi]ab^2r^2\sin\theta d\theta d\phi dr$$

simplifying this gives

$$Q_{zz} = \rho(r)\int_0^{2\pi}\int_0^{\pi}\int_0^{1}[2a^2r^2\cos^2\theta - b^2r^2\sin^2\theta]ab^2r^2\sin\theta d\theta d\phi dr$$

and using Mathematica yields

$$Q_{zz} = \rho(r)\frac{8\pi}{15}ab^2(a^2-b^2)$$

but we know that the charge density is given as

$$\rho(r) = \frac{q}{V} = \frac{3q}{4\pi ab^2} \qquad V_{ell} = \frac{4}{3}\pi ab^2$$

subsituting this expression gives

$$Q_{zz} = \frac{2}{5}q((a-b)(a+b))$$

solving for $a-b/R$ gives

$$\frac{a-b}{R} = \frac{5Q_{zz}}{2Ze(a+b)R} = \frac{5Q_{zz}}{4ZeR^2}$$

since we know that

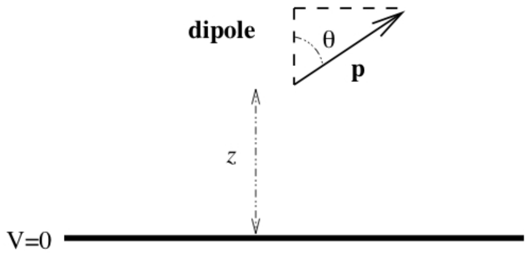
$$Q_{zz} = Qe$$

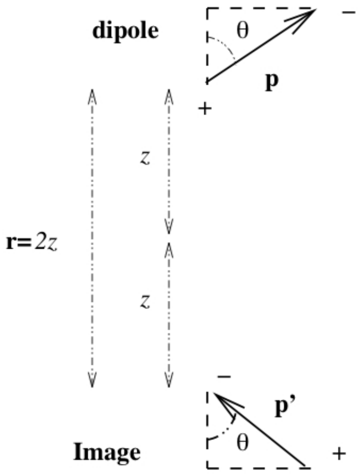
we find that the fractional difference is given by

$$\frac{a-b}{R} = \boxed{\frac{5Q}{4ZR^2} \approx 0.01}$$

Problem # 8

Consider a dipole with moment **p** at a distance z_0 above a grounded conducting plane that is taken to define the $x-y$ plane. **p** is taken to lie in the $x-z$ plane, oriented at an angle θ to the normal (**z**) of the plane. Find the torque **N** on the dipole. What are the equilibrium values of θ ? [Answer: **N** $=-p^2\sin\theta\cos\theta\hat{\mathbf{y}}/32\pi\varepsilon_0 z_0^3$)]





We know that the torque from an electric field on a dipole is given as

$$\mathbf{N} = \mathbf{p} \times \mathbf{E}$$

and we also know that the energy of a dipole is given as

$$\mathbf{E}_{dip} = \frac{1}{4\pi\varepsilon}\frac{1}{r^3}[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}]$$

we can solve this problem by using the image of the dipole, we know that the components of the dipole and the image are given by

$$\begin{aligned}
\mathbf{p}_{dip} &= \langle p\sin\theta, 0, p\cos\theta\rangle \\
\mathbf{p}_{im} &= \langle p'\sin\theta, 0, -p'\cos\theta\rangle \\
\mathbf{p}_{im} &= -\mathbf{p}_{dip}
\end{aligned}$$

we also know that $\hat{\mathbf{r}}$ points in the $\hat{\mathbf{z}}$ direction, thus

$$\mathbf{p}\cdot\hat{\mathbf{r}} = \mathbf{p}\cdot\hat{\mathbf{z}} = p'\cos\theta \qquad r = 2z_0$$

putting this into the electric dipole equation yields

$$\mathbf{E}_{im} = \frac{1}{4\pi\varepsilon}\frac{1}{(2z_0)^3}[3p'\cos\theta\hat{\mathbf{z}} + [p'\sin\theta\hat{\mathbf{x}} - p'\cos\theta\hat{\mathbf{z}}]] = \frac{1}{32\pi\varepsilon}\frac{1}{z_0^3}[p'\sin\theta\hat{\mathbf{x}} + 2p'\cos\theta\hat{\mathbf{z}}]$$

and thus we know that the torque is

$$\mathbf{N} = \mathbf{p}\times\mathbf{E} = \frac{p^2}{32\pi\varepsilon z_0^3}\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \sin\theta & 0 & \cos\theta \\ \sin\theta & 0 & 2\cos\theta \end{vmatrix}$$

which yields

$$\mathbf{N} = \boxed{-\frac{p^2\sin\theta\cos\theta}{32\pi\varepsilon z_0^3}\hat{\mathbf{y}}}$$

The equilibrium values of θ are given when the torque goes to zero, i.e

$$\boxed{\theta = \frac{n\pi}{2} \qquad n \geq 0 \ \ \text{integer}}$$

Problem # 9

Using Eq. 3.103, calculate the average electric field of a dipole, over a spherical volume of radius R, centered at the origin. Do the angular intervals first. [$Note$: You must express $\hat{\mathbf{r}}$ and $\hat{\theta}$ in terms of $\hat{\mathbf{x}}, \hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ before integrating. Compare your answer with the general theorem Eq. 3.105. The discrepency here is related to the fact that the field of the dipole blows up at $r = 0$. The angular integral is zero, but th radial integral is infinite, so we really don't know what to make of the answer. To resolve this dilemma, let's say Eq. 3.103 applies *outside a tiny sphere of radius* ε- its contribution to E_{ave} is then *unambiguously* zero, and the whole answer has to come from the field *inside* the $\varepsilon-$sphere.

We know that the average electric field and the electric field from the dipole can be expressed as

$$\mathbf{E}_{ave} = \frac{\int_{space} \mathbf{E}_{dip} d\tau}{V} \qquad \mathbf{E}_{dip} = \frac{p}{4\pi\varepsilon}(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\theta})$$

we need to change to cartesian coordiantes so that we can evaluate this integral, i.e

$$\begin{aligned} \hat{\mathbf{r}} &= \sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}} \\ \hat{\theta} &= \cos\theta\cos\phi\hat{\mathbf{x}} + \cos\theta\sin\phi\hat{\mathbf{y}} - \sin\theta\hat{\mathbf{z}} \end{aligned}$$

thus $\mathbf{E}_{dip}$ in cartesian coordinates gives

$$\mathbf{E}_{dip} = \frac{p}{4\pi\varepsilon}[3\sin\theta\cos\theta\cos\phi\hat{\mathbf{x}} + 3\sin\theta\cos\theta\sin\phi\hat{\mathbf{y}} + (2\cos^2\theta - \sin^2\theta)\hat{\mathbf{z}}]$$

and so we find that the integral can be expressed as

$$\begin{aligned} \int_{space} \mathbf{E}_{dip} d\tau &= \frac{p}{4\pi\varepsilon}\int_0^r\int_0^{2\pi}\int_0^{\pi} \frac{3\sin^2\theta\cos\theta\cos\phi}{r} dr d\theta d\phi\hat{\mathbf{x}} + \frac{p}{4\pi\varepsilon}\int_0^r\int_0^{2\pi}\int_0^{\pi} \frac{3\sin^2\theta\cos\theta\sin\phi}{r} dr d\theta d\phi\hat{\mathbf{y}} \\ &+ \frac{p}{4\pi\varepsilon}\int_0^r\int_0^{2\pi}\int_0^{\pi} \frac{(2\cos^2\theta - \sin^2\theta}{r} dr d\theta d\phi\hat{\mathbf{y}} \end{aligned}$$

using Mathematica yields (for the angular integrals)

$$\int_{ang} \mathbf{E}_{dip} d\tau = 0$$

and we know that the integral for the radial component (at the origin) yields

$$\int_{rad} \mathbf{E}_{dip} d\tau = \infty$$

thus we find that

$$\mathbf{E}_{ave} = 0 \cdot \infty$$

which seems very weird? There must be a delta function in the center of the sphere.

(b). What must the field *inside* the $\varepsilon-$sphere be, in order for the general theorem (3.105) to hold? [$Hint$: since ε is arbitrarily small, we're talking about something that is infinite at $r = 0$ and whose integral over an infinitesimal volume is finite.][$Answer$: $-(\mathbf{p}/3\varepsilon_0)\delta^3(\mathbf{r})$]

[Evidently, the *true* field of a dipole is

$$\mathbf{E}_{dip}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}] - \frac{1}{3\varepsilon_0}\mathbf{p}\delta^3(\mathbf{r})$$

You may well wonder how we missed the delta-function term when we calculated the field back in Sect. 3.4.4. The answer is that the differentiation leading to Eq. 3.103 is perfectly valid *except* at $r = 0$, but we should have known (from our experience in Sect 1.5.1) that the point $r = 0$ is problematic.

Since we know that the average electric field inside the sphere is not zero we can write the electric dipole as

$$\mathbf{E}_{dip} = \frac{1}{4\pi\varepsilon r^3}[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}] + K\delta^3(r)$$

where K is a non-zero coefficient that we need to solve for. we can solve for K by solving the average electric field in terms of a delta function

$$\mathbf{E}_{ave} = \frac{\int K\delta^3(r)d\tau}{V} = \frac{3K}{4\pi R^3}$$

setting this expression to equation 3.105 we find

$$\frac{3K}{4\pi R^3} = -\frac{\mathbf{p}}{4\pi\varepsilon_0 R^3} \quad K = -\frac{\mathbf{p}}{3\varepsilon_0}$$

thus the correct expression for the electric dipole is given by

$$\mathbf{E}_{dip} = \boxed{\frac{1}{4\pi\varepsilon r^3}[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}] - \frac{\mathbf{p}}{3\varepsilon_0}\delta^3(r)}$$

Chapter 4

Electric Fields in Matter

4.1 Problems and Solutions

Problem # 1 Forces on dipoles

(a). A dipole $\mathbf{p}$ with fixed magnitude is placed in an external electric field $\mathbf{E}(r)$ that varies with position. Show that there is a force on the dipole, and find how this force depends on the orientation of the dipole and on the variation of the field. [Answer: $\mathbf{F} = (\mathbf{p}\cdot\nabla)\mathbf{E}$]

So we know that

$$\mathbf{F}_{+} = q\mathbf{E}_{+} \qquad \mathbf{F}_{-} = -q\mathbf{E}_{-}$$

and we also know that the total force is given by

$$\mathbf{F}_{tot} = \mathbf{F}_{+} + \mathbf{F}_{-} = q(\mathbf{E}_{+} - \mathbf{E}_{-}) = q\Delta\mathbf{E}$$

where the components of E are given by

$$\Delta E_x = (\nabla E_x)\cdot\mathbf{d} \quad \Delta E_y = (\nabla E_y)\cdot\mathbf{d} \quad \Delta E_z = (\nabla E_z)\cdot\mathbf{d}$$

where $\mathbf{d}$ is the displacement vector. Thus we know

$$\Delta\mathbf{E} = (\mathbf{d}\cdot\nabla)\mathbf{E}$$

thus the force is given by

$$\boxed{\mathbf{F} = (\mathbf{p}\cdot\nabla)\mathbf{E}}$$

(b). Find the force and torque on a dipole in the field of a point charge. Let the charge q be at the origin, and the dipole $\mathbf{p} = p(\sin\xi_0\hat{\mathbf{x}} + \cos\xi_0\hat{\mathbf{z}})$ be at the point (0, 0, z_0). Also, find the force on q, and verify Newton's third law. [Answer: the force on q is

$$\mathbf{F} = \frac{p_0 q}{4\pi\varepsilon_0 z_0^3}(-\sin\xi_0\hat{\mathbf{x}} + 2\cos\xi_0\hat{\mathbf{z}})$$

We know that the potential of a point charge can be written as (in cartesian coordinates)

$$V_c = \frac{q}{4\pi\varepsilon}\frac{1}{\sqrt{x^2+y^2+z^2}}$$

and we know that the electric field can be written as

$$\mathbf{E} = -\nabla V = \frac{q}{4\pi\varepsilon}\left[\frac{x}{(x^2+y^2+z^2)^{3/2}}\hat{\mathbf{x}} + \frac{y}{(x^2+y^2+z^2)^{3/2}}\hat{\mathbf{y}} + \frac{z}{(x^2+y^2+z^2)^{3/2}}\hat{\mathbf{z}}\right]$$

now that we have the electric field we can find the force

$$\mathbf{F} = \left(p\sin\xi_0\frac{\partial}{\partial x}\mathbf{E} + p\cos\xi_0\frac{\partial}{\partial z}\mathbf{E}\right)$$

which yields

$$\mathbf{F}(0,0,z_0) = \boxed{\frac{pq}{4\pi\varepsilon}[\sin\xi_0\hat{\mathbf{x}} - 2\cos\xi_0\hat{\mathbf{z}}]}$$

and the torque is given by

$$\mathbf{N}(0,0,z_0) = \mathbf{p}\times\mathbf{E} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ p\sin\xi_0 & 0 & p\cos\xi_0 \\ 0 & 0 & \frac{q}{4\pi\varepsilon z_0^2} \end{vmatrix} = \boxed{-\frac{pq\sin\xi_0}{4\pi\varepsilon z_0^2}\hat{\mathbf{y}}}$$

To verify Newton's third law we need to show the force on q from the dipole is equal and opposite of the force on the dipole from q, thus we can start with the defenition for the electric field of a dipole

$$\mathbf{E}_{dip}(r) = \frac{1}{4\pi\varepsilon r^3}[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}]$$

and we know that the force is given by

$$\mathbf{F} = q\mathbf{E}_{dip}$$

we also know that

$$\hat{\mathbf{r}} = -\hat{\mathbf{z}} \qquad \mathbf{p} = p(\sin\xi_0\hat{\mathbf{x}} + \cos\xi_0\hat{\mathbf{z}}) \qquad (\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} = p\cos\xi_0\hat{\mathbf{z}}$$

thus we can see that the force is

$$\mathbf{F} = \frac{pq}{4\pi\varepsilon z_0^3}[3\cos\xi_0\hat{\mathbf{z}} - \sin\xi_0\hat{\mathbf{x}} - \cos\xi_0\hat{\mathbf{z}}] = \boxed{\frac{pq}{4\pi\varepsilon z_0^3}[2\cos\xi_0\hat{\mathbf{z}} - \sin\xi_0\hat{\mathbf{x}}]}$$

thus we can see that

$$\mathbf{F} = q\mathbf{E}_{dip} = -q\mathbf{E}_c$$

We know that the equilibrium values for θ are given when

$$\boxed{\theta = \frac{n\pi}{2} \qquad n \geq 0}$$

Problem # 2 Capacitors with dielectric slabs

The space between the plates of a parallel capacitor (Figure 1) is filled with two slabs of linear dialectric material. Each slab has thickness a, so the total distance between the plates is $2a$. Slab 1 has a dialectric constant of 2, and Slab 2 has a dialectric constant of 1.5. The free charge density on the top plate is σ and onthe bottom plate $-\sigma$.

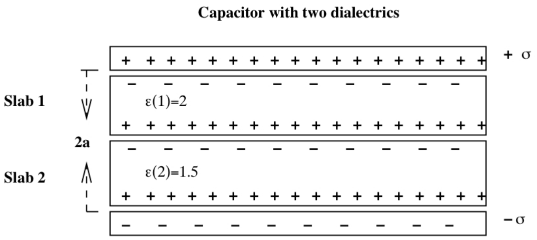


Figure 1

a). Find the electric displacement **D** in each slab.

Since we know what the free charge is on both the top plate of the capacitor and the bottom plate of the capacitor we can apply the integral form of Gauss's law to find the displacement, using

$$\oint_{top} \mathbf{D} \cdot d\mathbf{a} = Q_f$$

we can find what the displacement if we place a Gaussian pillbox encompassing the top plate of the capacitor and Slab 1

$$Q_f = \sigma_f A \qquad \oint_{top} \mathbf{D}_1 \cdot d\mathbf{a} = \sigma_f A \qquad d\mathbf{a} = A(\hat{\mathbf{z}})$$

where $\hat{\mathbf{z}}$ points in the direction of the electric field, which is downward. So for Slab 1 we find the displacement to be

$$\mathbf{D}_1 = \sigma_f(\hat{\mathbf{z}}) = \sigma(\hat{\mathbf{z}})$$

and for the displacement of Slab 2 we can place a Gaussian pillbox encompassing the bottom plate of the capacitor and Slab 2

$$Q_f = \sigma_f A \qquad \oint_{bot} \mathbf{D}_2 \cdot d\mathbf{a} = \sigma_f A \qquad d\mathbf{a} = A(\hat{\mathbf{z}})$$

thus

$$\mathbf{D}_2 = \sigma_f(\hat{\mathbf{z}}) = -\sigma(\hat{\mathbf{z}})$$

and the magnitudes are given by

$$\boxed{D_1 = D_2 = \sigma}$$

b). Find the electric field **E** in each slab.

given that we know what **D** is in each slab we can easily find **E**

$$\mathbf{E}_1 = \frac{\mathbf{D}_1}{\varepsilon_1} \quad \varepsilon_1 = 2\varepsilon_0 \qquad \mathbf{E}_1 = \frac{\sigma}{2\varepsilon_0}\hat{\mathbf{z}}$$

and

$$\mathbf{E}_2 = \frac{\mathbf{D}_2}{\varepsilon_2} \quad \varepsilon_2 = \frac{3}{2}\varepsilon_0 \qquad \mathbf{E}_2 = -\frac{2\sigma}{3\varepsilon_0}\hat{\mathbf{z}}$$

and the magnitudes are given by

$$\boxed{E_1 = \frac{\sigma}{2\varepsilon_0} \qquad E_2 = \frac{2\sigma}{3\varepsilon_0}}$$

c). Find the polarization $\mathbf{P}$ in each slab.

We know that the polarization is defined as

$$\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E}$$

where χ_e is defined as

$$\frac{\varepsilon}{\varepsilon_0} = \varepsilon_r = \varepsilon_0(1+\chi_e) \quad \Rightarrow \quad \chi_e = \varepsilon_r - 1 \qquad \chi_1 = 1 \qquad \chi_2 = \frac{1}{2}$$

for Slab 1 and 2 we find

$$\mathbf{P}_1 = \frac{\sigma}{2}\hat{\mathbf{z}} \qquad \mathbf{P}_2 = -\frac{\sigma}{3}\hat{\mathbf{z}}$$

and the magnitudes are given by

$$\boxed{P_1 = \frac{\sigma}{2} \qquad P_2 = \frac{\sigma}{3}}$$

d). Find the potential difference between the plates.

we know that the potential difference is given by the following integral

$$\Delta V = |V_1 - V_2| = |\int_0^a \mathbf{E}_1 \cdot d\mathbf{l} - \int_a^{2a} \mathbf{E}_2 \cdot d\mathbf{l}| = \left| \frac{\sigma a}{2\varepsilon_0} + \frac{2\sigma a}{3\varepsilon_0} \right|$$

given that $d\mathbf{l} = a\hat{\mathbf{z}}$ we find

$$\Delta V = \boxed{\frac{7\sigma a}{6\varepsilon_0}}$$

e). Find the location and amount of all bound charge.

given that we know the polarization we can find what the bound charge is by using

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}$$

where $\hat{\mathbf{n}}$ points outward with respect to the dielectric, so for Slab 1 we find

$$\begin{aligned} \sigma_b &= -\frac{\sigma}{2} && \text{Slab 1 top surface} \\ \sigma_b &= \frac{\sigma}{2} && \text{Slab 1 bottom surface} \end{aligned}$$

and for Slab 2 we find

$$\begin{aligned} \sigma_b &= -\frac{\sigma}{3} && \text{Slab 2 top surface} \\ \sigma_b &= \frac{\sigma}{3} && \text{Slab 2 bottom surface} \end{aligned}$$

this is illustrated in Figure 2.

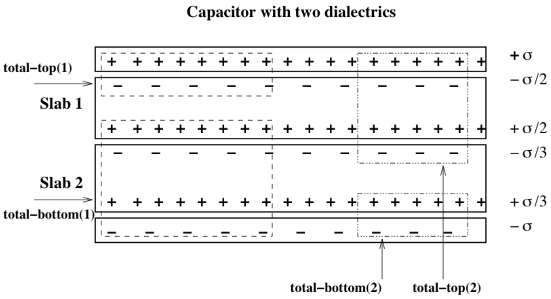


Figure 2

f). Now that you know all the charge (free and bound), recalculate the field in each slab, and confirm your answer to **b**.

Using Figure 2 we can find the total charge σ_t for Slab 1 and Slab 2. For the top of Slab 1 we add up all the charge in the dashed box labeled total-top(1) which yields

$$\sigma_t = \sigma - \frac{\sigma}{2} = \frac{\sigma}{2} \qquad \text{Top of Slab 1}$$

for the bottom of Slab 1 we add up all the charge in the box labeled total-bottom(1) which yields

$$\sigma_t = \frac{\sigma}{2} - \frac{\sigma}{3} + \frac{\sigma}{3} - \sigma = -\frac{\sigma}{2} \qquad \text{bottom of Slab 1}$$

and now for the bottom of Slab 2 we add up all the charge in the dashed-dot box labeled total-bottom(2) which yeilds

$$\sigma_t = \frac{\sigma}{3} - \sigma = -\frac{2}{3}\sigma \quad \text{bottom of Slab 2}$$

and finally for the top of Slab 2 we add up all the charge in the dashed-dot box labeled total-top(2) which yields

$$\sigma_t = -\frac{\sigma}{3} + \frac{\sigma}{2} - \frac{\sigma}{2} + \sigma = \frac{2}{3}\sigma \quad \text{Top of Slab 2}$$

and the magnitude of the electric field is given by

$$E_1 = \frac{\sigma_t}{\varepsilon_0} = \frac{\sigma}{2\varepsilon_0} \qquad E_2 = \frac{\sigma_t}{\varepsilon_0} = \frac{2\sigma}{3\varepsilon_0}$$

which is the same thing we found in part **b**.

part 2: problem 2

Suppose you have enough linear dielectric material, of dielectric constant ε_r , to *half*-fill a parallel-plate capacitor (Figure 3). By what fraction is the capacitance increased when you distribute the material as in Fig.3 (a)? How about Fig.3 (b)?

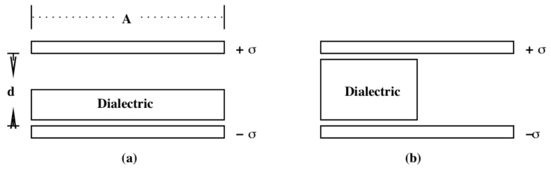


Figure 3

For Figure 3 **(a)** we can see that the displacement is given by

$$\oint \mathbf{D} \cdot d\mathbf{a} = Q_f = \sigma A \qquad D = \sigma$$

this is the displacement inside the dielectric, with this we can now find what the electric field is inside the dielectric

$$\mathbf{E} = \frac{\mathbf{D}}{\varepsilon} \qquad E_{dia} = \frac{\sigma}{\varepsilon}$$

we also know that the electric field between the dielectric and the top plate of the capacitor is given by

$$E_{air} = \frac{\sigma}{\varepsilon_0}$$

given that the defenition of capacitance is given by

$$C_{1/2} = \frac{Q}{V_{1/2}}$$

we are now in a position to find what the potential is between the capacitor

$$V_{1/2} = \int_0^{d/2} \mathbf{E}_{air} \cdot d\mathbf{l} + \int_{d/2}^{d} \mathbf{E}_{dia} \cdot d\mathbf{l} = \frac{\sigma d}{2\varepsilon_0} + \frac{\sigma d}{2\varepsilon} = \frac{\sigma d}{2}\left(\frac{1}{\varepsilon_0} + \frac{1}{\varepsilon}\right) = \frac{\sigma d}{2\varepsilon}(\varepsilon_r + 1)$$

thus the capacitance is given by

$$C_{1/2} = \frac{Q}{V_{1/2}} = \frac{\sigma A}{V_{1/2}} = \frac{2A\varepsilon}{d(\varepsilon_r + 1)}$$

and since we know that a capacitor with no dielectric has a capacitance given by

$$C = \frac{A\varepsilon_0}{d}$$

thus the ratio of the capacitance with dielectric to capacitance without dielectric for Figure 3 **(a)** is given by

$$\frac{C_{1/2}}{C} = \frac{2\varepsilon}{\varepsilon_0(\varepsilon_r + 1)} = \boxed{\frac{2\varepsilon_r}{(\varepsilon_r + 1)}}$$

From Figure 3 **(b)** we can see that the area has been decreased by 1/2 and the dielectric now fills this entire 1/2-space, thus we can treat this as two separate capacitors one in which the 1/2 area is filled with the dielectric and another 1/2 area that has no dielectric. We find that the total capacitance is given by

$$C_{1/2} = \frac{A\varepsilon}{2d} + \frac{A\varepsilon_0}{2d} = \frac{A}{2d}(\varepsilon + \varepsilon_0)$$

and we know that the capacitance with no dielectric is given by

$$C = \frac{A\varepsilon_0}{d}$$

thus the ratio gives

$$\frac{C_{1/2}}{C} = \frac{\varepsilon + \varepsilon_0}{2\varepsilon_0} = \boxed{\frac{\varepsilon_r + 1}{2}}$$

Problem # 3

A spherical conductor, of radius a, carries a charge Q. It is sorrounded by linear dielectric material of susceptibility χ_e, out to radius b. Find the energy of this configuration.

We know that the total work done, as we build up the free charge from zero to its final confiuguartion is given by

$$W = \frac{1}{2}\int \mathbf{D} \cdot \mathbf{E} d\tau$$

We can find **D** by using

$$\oint \mathbf{D}\cdot d\mathbf{a} = Q_f \qquad \mathbf{D} = \frac{Q}{4\pi r^2}\hat{\mathbf{r}} \qquad r > a$$

we know that **P=D=E=0** inside the sphere. Now that we know **D** we can easily find **E**,

$$\mathbf{E} = \frac{\mathbf{D}}{\varepsilon} = \frac{Q}{4\pi\varepsilon r^2}\hat{\mathbf{r}} \qquad a < r < b$$

and

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}} \qquad r > b$$

now that we have **D** and **E** we can now find what the energy of this configuration is

$$W = \frac{Q^2}{2}\left[\int_a^b \frac{1}{(4\pi)^2\varepsilon r^2} dr d\Omega + \int_b^\infty \frac{1}{(4\pi)^2\varepsilon_0 r^2} dr d\Omega\right]$$

knowing that $d\Omega = 4\pi$ when integrating over all solid angles allow us to symplify this as

$$W = \frac{Q^2}{8\pi}\left[\frac{1}{\varepsilon}\int_a^b \frac{1}{r^2}dr + \frac{1}{\varepsilon_0}\int_b^\infty \frac{1}{r^2}dr\right] = \frac{Q^2}{8\pi}\left[\frac{1}{\varepsilon}\left[\frac{1}{a}-\frac{1}{b}\right] + \frac{1}{\varepsilon_0}\left[\frac{1}{b}\right]\right]$$

we can also write this as

$$W = \frac{Q^2}{8\pi\varepsilon}\left[\left(\frac{1}{a}-\frac{1}{b}\right) + \varepsilon_r\frac{1}{b}\right] = \frac{Q^2}{8\pi\varepsilon}\left[\frac{1}{a} + (\varepsilon_r - 1)\frac{1}{b}\right]$$

but since we know

$$\varepsilon_r = 1 + \chi_e \qquad \varepsilon = \varepsilon_0(1+\chi_e)$$

we find

$$W = \boxed{\frac{Q^2}{8\pi\varepsilon_0(1+\chi_e)}\left[\frac{1}{a} + \chi_e\frac{1}{b}\right]}$$

Problem # 4

The dielectric constant of air is 1.00059. From this, determine the mean polarizability of atmospheric molecules. Compare this result to the atomic polarizabilities listed, for example, in Griffiths Table 4.1.

We can write the polarization as

$$\mathbf{P} = \chi_e\varepsilon_0\mathbf{E}$$

The polarization in a dialectric is also defined as

$$\mathbf{P} = n\mathbf{p}$$

where **p** is the dipole moment and n is the particle density. We also know that the dipole moment can be defined as

$$\mathbf{p} = \alpha\mathbf{E}_{other}$$

where α is the polarizability and $\mathbf{E}_{other}$ is the total electric field, the applied field and the enduced field

$$\mathbf{E}_{other} = \mathbf{E} - \mathbf{E}_{self}$$

we need to find out what the self induced field is. The external field produced by the sphere itself is a pure dipole. The total field inside the conducting sphere is 0, because a conductor is an equipotential, thus the induced field inside must be $-\mathbf{E}_0$, this field is needed to cancel the applied uniform field, so we find

$$\mathbf{E}_{self} = -E_0$$

If we treat the sphere as a conducting sphere we find that the dipole moment goes as

$$\mathbf{p} = 4\pi\varepsilon_0 a^3 E_0$$

where a is the radius, thus $\mathbf{E}_{self}$ can be written as

$$\mathbf{E}_{self} = -\frac{\mathbf{p}}{4\pi\varepsilon_0 a^3}$$

substituting this in $\mathbf{E}_{other}$ we find

$$\mathbf{E}_{other} = \mathbf{E} + \frac{\mathbf{p}}{4\pi\varepsilon_0 a^3}$$

we also know that

$$\mathbf{E}_{self} = -\frac{n\mathbf{p}}{3\varepsilon_0} = -\frac{\mathbf{P}}{3\varepsilon_0}$$

thus we find

$$\mathbf{P} = n\mathbf{p} = n\alpha\left(\mathbf{E} + \frac{\mathbf{P}}{3\varepsilon_0}\right)$$

there is a linear relationship between $\mathbf{P}$ and $\mathbf{E}$, and the susceptibility which is given by

$$\chi_e = \frac{n\alpha/\varepsilon_0}{1 - n\alpha/(3\varepsilon_0)}$$

if we solve for the atomic polarizability we find

$$\alpha = \frac{\varepsilon_0}{n}\frac{\chi_e}{1+\chi_e/3} = \frac{3\varepsilon_0}{n}\frac{\kappa-1}{\kappa+2}$$

and usin the ideal gas law we find

$$n = \frac{N}{V} = \frac{kT}{P}$$

we find for $\alpha/4\pi\varepsilon_0$

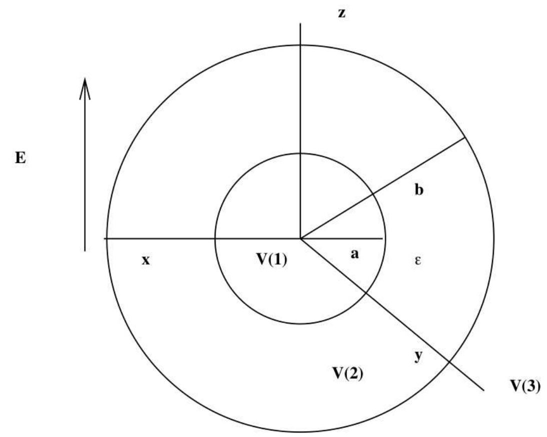
$$\frac{\alpha}{4\pi\varepsilon_0} = \frac{3kT}{4\pi P}\frac{\kappa-1}{\kappa+2} \approx \boxed{1.90\times10^{-30}\frac{1}{\mathrm{m}^3}}$$

using $T = 293\,\mathrm{K}$ and $1\times10^5\mathrm{Pa}$ for the pressure, along with the dielectric constant given.

Problem # 5

A hollow dielectric sphere, with dielectric constant $\varepsilon/\varepsilon_0 = \kappa$, inner radius a and outer radius b, is placed in a uniform applied electric field $E_0\hat{\mathbf{z}}$. The presence of the sphere changes the field. Find the field in the three defined regions, i.e. $r < a, a < r < b$ and $r > b$. What is the field at the center of the spherical shell? What is the dipole moment of the dielectric medium? [for the last part, youÕll find:

$$p_z = \frac{4\pi\varepsilon_0(a^3-b^3)b^3E_0(1+2\kappa)(\kappa+1)}{2a^3(\kappa-1)^2 - b^3(2+\kappa)(1+2\kappa)}$$



Using the general solution for the potential in spherical coordinates we find thet the potentials are given by

$$\begin{aligned} V_1 &= \sum_{l=0} A_l r^l P_l \cos\theta \qquad r<a \\ V_2 &= \sum_{l=0} \left(B_l r^l + \frac{C_l}{r^{l+1}} \right) P_l \cos\theta \quad a<r<b \\ V_3 &= \sum_{l=0} \frac{D_l}{r^{l+1}} P_l \cos\theta - E_0 r \cos\theta \end{aligned}$$

and the boundary conditions are given by

$$\begin{aligned} (i)\ \ V_1 &= V_2 \qquad r=a \\ (ii)\ \ V_2 &= V_3 \qquad r=b \\ (iii)\ \varepsilon \frac{\partial V_2}{\partial r} &= \varepsilon_0 \frac{\partial V_1}{\partial r} \quad r=a \\ (iv)\ \ \varepsilon \frac{\partial V_2}{\partial r} &= \varepsilon_0 \frac{\partial V_3}{\partial r} \quad r=b \end{aligned}$$

thus we have four unknown constants and four boundary conditions. Lets figure out what these constants are using the boundary conditions. From 1 we find

$$\sum_{l=0} A_l a^l P_l \cos\theta = \sum_{l=0} \left(B_l a^l + \frac{C_l}{a^{l+1}} \right) P_l \cos\theta$$

which simplifies to

$$A_l a^l = B_l a^l + \frac{C_l}{a^{l+1}} \quad l \neq 1 \qquad \boxed{A = B + \frac{C}{a^3} \quad l=1}$$

if

$$l \neq 1 \quad A_l = B_l = C_l = 0$$

from this and from the next boundary condition we know that $l=1$ for all cases. From the second boundary condition we find (using $l=1$)

$$\boxed{Bb^3 + C = D - E_0 b^3}$$

from boundary condition number 3 we find

$$\varepsilon_0 A = \varepsilon\left[B - \frac{2C}{a^3}\right] \qquad \kappa = \frac{\varepsilon}{\varepsilon_0}$$

which simplifies to

$$\boxed{A = \kappa B - \frac{2\kappa C}{a^3}}$$

and from our last boundary condition we find

$$\boxed{-\frac{2D}{b^3} - E_0 = \kappa B - \frac{2\kappa C}{b^3}}$$

so now we have four equation and four unknowns, we can do this by brute force or just use use Mathematica. The four equations are summarized as

$$\begin{aligned} A &= B + \frac{C}{a^3} \\ Bb^3 + C &= D - E_0 b^3 \\ A &= \kappa B - \frac{2\kappa C}{a^3} \\ -\frac{2D}{b^3} - E_0 &= \kappa B - \frac{2\kappa C}{b^3} \end{aligned}$$

Mathematica yields

$$\begin{aligned} A &= \frac{9b^3 E_0 \kappa}{2a^3(\kappa-1)^2 - b^3(2+\kappa)(1+2\kappa)} \\ B &= \frac{3b^3 E_0(1+2\kappa)}{2a^3(\kappa-1)^2 - b^3(2+\kappa)(1+2\kappa)} \\ C &= \frac{3a^3 b^3 E_0(\kappa-1)}{2a^3(\kappa-1)^2 - b^3(2+\kappa)(1+2\kappa)} \\ D &= \frac{(a^3-b^3)b^3 E_0(1+2\kappa)(\kappa-1)}{2a^3(\kappa-1)^2 - b^3(2+\kappa)(1+2\kappa)} \end{aligned}$$

plugging these constants into the general expression for all potentials yield

$$V_1 = \frac{9b^3 E_0 \kappa}{2a^3(\kappa-1)^2 - b^3(2+\kappa)(1+2\kappa)} r\cos\theta = \frac{9b^3 E_0 \kappa}{2a^3(\kappa-1)^2 - b^3(2+\kappa)(1+2\kappa)} z \qquad r\cos\theta$$

and for the other two we find

$$V_2 = \frac{3b^3 E_0(1+2\kappa)}{2a^3(\kappa-1)^2 - b^3(2+\kappa)(1+2\kappa)} z + \frac{3a^3 b^3 E_0(\kappa-1)}{2a^3(\kappa-1)^2 - b^3(2+\kappa)(1+2\kappa)} \frac{z}{(x^2+y^2+z^2)^{3/2}}$$

and the final potential is given by

$$V_3 = \frac{(a^3-b^3)b^3 E_0(1+2\kappa)(\kappa-1)}{2a^3(\kappa-1)^2 - b^3(2+\kappa)(1+2\kappa)} \frac{z}{(x^2+y^2+z^2)^{3/2}} - E_0 z$$

The electric field in all three regions is given by

$$E = -\nabla V$$

so

$$\boxed{E_1 = -\frac{9b^3E_0\kappa}{2a^3(\kappa-1)^2 - b^3(2+\kappa)(1+2\kappa)}\hat{\mathbf{z}} \qquad r < a}$$

the electric field inside the dialectric is given by

$$\begin{aligned} E_2 &= -\frac{3b^3E_0(1+2\kappa)}{2a^3(\kappa-1)^2 - b^3(2+\kappa)(1+2\kappa)}\hat{\mathbf{z}} + \frac{3a^3b^3E_0(\kappa-1)}{2a^3(\kappa-1)^2 - b^3(2+\kappa)(1+2\kappa)} \times \\ & \left[-\frac{3xz}{(x^2+y^2+z^2)^{5/2}}\hat{\mathbf{x}} - \frac{3yz}{(x^2+y^2+z^2)^{5/2}}\hat{\mathbf{y}} - \frac{3z^2}{(x^2+y^2+z^2)^{5/2}}\hat{\mathbf{z}} + \frac{1}{(x^2+y^2+z^2)^{3/2}}\hat{\mathbf{z}}\right] \end{aligned}$$

and finally for the electric field on the outside we find, using

$$K = \frac{(a^3-b^3)b^3E_0(1+2\kappa)(\kappa-1)}{2a^3(\kappa-1)^2 - b^3(2+\kappa)(1+2\kappa)}$$

$$E_3 = K\left[-\frac{3xz}{(x^2+y^2+z^2)^{5/2}}\hat{\mathbf{x}} - \frac{3yz}{(x^2+y^2+z^2)^{5/2}}\hat{\mathbf{y}} - \frac{3z^2}{(x^2+y^2+z^2)^{5/2}}\hat{\mathbf{z}} + \frac{1}{(x^2+y^2+z^2)^{3/2}}\hat{\mathbf{z}}\right] - E_0\hat{\mathbf{z}}$$

The field in the center of the dialectric is given by

$$E_{center} = \boxed{-\frac{9b^3E_0\kappa}{2a^3(\kappa-1)^2 - b^3(2+\kappa)(1+2\kappa)}\hat{\mathbf{z}}}$$

We know that the potential in terms of the dipole moment is given by

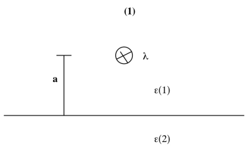
$$V = \frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{4\pi\varepsilon_0 r^2} = \frac{p_z\cos\theta}{4\pi\varepsilon_0 r^2}$$

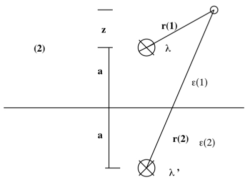
and looking at the potential on the outside we can see that the potential from the dipole moment is just the first term in the potential V_3, the E_0 is from the applied field, we are only interested in the potential from the dipole, i.e

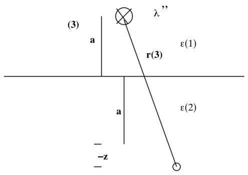
$$p_z = \boxed{\frac{4\pi\varepsilon_0(a^3-b^3)b^3E_0(1+2\kappa)(\kappa-1)}{2a^3(\kappa-1)^2 - b^3(2+\kappa)(1+2\kappa)}}$$

Problem # 6

Consider two insulating media with dielectric constants $\varepsilon_1/\varepsilon_0 = \kappa_1$ and $\varepsilon_2/\varepsilon_0 = \kappa_2$, placed together with a planar interface between them. In the region of ?1 there is an infinite line charge with linear charge density κ_1, parallel to the interface at perpendicular distance







a. Use the method of images to find the electric field in both media.

We can see from diagram (2) that the potential in the top region is given by

$$V_t = \frac{\lambda}{2\pi\varepsilon_1}\ln|z-a| + \frac{\lambda'}{2\pi\varepsilon_1}\ln|z+a|$$

and the potential at the bottom is given by

$$V_b = \frac{\lambda''}{2\pi\varepsilon_2}\ln|z-a|$$

we also know that the boundary conditions are given by

$$V_t = V_b \qquad z = 0$$

from continuity. Our othert boundary condition is given as

$$\varepsilon_1\frac{\partial V_t}{\partial z} = \varepsilon_2\frac{\partial V_b}{\partial z}$$

From boundary condition number 1 we can see that

$$\lambda + \lambda' = \frac{\varepsilon_1}{\varepsilon_2}\lambda''$$

and from condition 2 we find

$$\lambda - \lambda' = \lambda''$$

plugging in λ'' from the last equation into the previous one yields

$$\lambda + \lambda' = \frac{\varepsilon_1}{\varepsilon_2}(\lambda - \lambda')$$

thus

$$\lambda' = \boxed{\lambda\left(\frac{\kappa_2 - \kappa_1}{\kappa_2 + \kappa_1}\right)}$$

plugging this into either equation yields

$$\lambda'' = \lambda\left[1 + \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2 + \varepsilon_1}\right] = \boxed{\left(\frac{2\kappa_2}{\kappa_2 + \kappa_1}\right)\lambda}$$

plugging these solutions into our potentials we find

$$V_t = \frac{\lambda}{2\pi\varepsilon_1}\ln|z-a| + \lambda\left(\frac{\kappa_2-\kappa_1}{\kappa_2+\kappa_1}\right)\frac{1}{2\pi\varepsilon_1}\ln|z+a|$$

and the electric field at the top is given by

$$E_t = \boxed{-\frac{\lambda}{2\pi\varepsilon_1|z-a|} - \lambda\left(\frac{\kappa_2-\kappa_1}{\kappa_2+\kappa_1}\right)\frac{1}{2\pi\varepsilon_1}\frac{1}{|z+a|}}$$

and the potential at the bottom is given by

$$V_b = \frac{1}{2\pi\varepsilon_2}\left(\frac{2\kappa_2}{\kappa_2+\kappa_1}\right)\lambda\ln|z-a|$$

and the electric field is given as

$$E_b = \boxed{-\frac{1}{\pi}\left(\frac{1}{\kappa_2+\kappa_1}\right)\lambda\frac{1}{|z-a|}}$$

Show that the force per unit length on the line charge is

$$\frac{dF}{dL} = \frac{\lambda^2(\kappa_1-\kappa_2)}{4\pi\varepsilon_1 a(\kappa_1-\kappa_2)}$$

We know that

$$F = qE \qquad dF = dqE$$

but

$$dq = \lambda dl$$

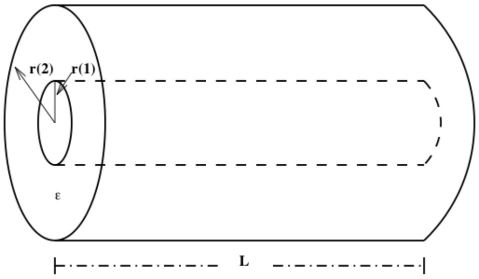
and since we know that there is no self induced force on λ we must use the λ' term as the one contributing to the force, so we find

$$\frac{dF}{dL} = \lambda E_{1\lambda'} = \boxed{\lambda^2\frac{(\kappa_1-\kappa_2)}{4\pi\varepsilon_1(\kappa_1+\kappa_2)}}$$

Note that the sign of the force depends on the difference $\kappa_1 - \kappa_2$. Explain physically what's going on here.

We can see from the expression for the force that if κ_1 is greater than κ_2 the force will be pointing in the $\hat{\mathbf{z}}$ direction and if we have the inverse it will point in the -$\hat{\mathbf{z}}$ direction.

Problem # 7



Two long cylindrical sheets of metal (radii r_1 and r_2 with $r_2 > r_1$) are arranged coaxially. The plates connected to a battery that maintains a potential difference V between the sheets. The region between the conductors is filled with a material of conductivity σ and permittivity ε.

a) Determine the capacitance per unit length of this system.

We know that the capacitance and the potential are given by

$$C = \frac{Q}{V} \qquad V = -\int \mathbf{E} \cdot d\mathbf{l}$$

we can find the electric field by using Gauss's law, i.e

$$\int \mathbf{E} \cdot d\mathbf{a} = \frac{Q}{\varepsilon} \qquad d\mathbf{a} = 2\pi r L \hat{\mathbf{r}}$$

we find the electric field to be given as

$$\mathbf{E} = \frac{Q}{2\pi r L \varepsilon} \hat{\mathbf{r}}$$

knowing the electric field allows us to find the potential

$$V = -\int_{r_2}^{r_1} \frac{Q}{2\pi r L \varepsilon} dr = \frac{Q}{2\pi L \varepsilon} \ln\left(\frac{r_2}{r_1}\right)$$

knowing this allows us to find the capacitance per unit length

$$\boxed{\frac{C}{L} = \frac{2\pi\varepsilon}{\ln\left(\frac{r_2}{r_1}\right)}}$$

b) Use Ohm's law to calculate the electric current per unit length between the conducting shells.

We know that Ohm's law can be expressed as

$$I = \int \mathbf{J} \cdot d\mathbf{a} = \sigma \int \mathbf{E} \cdot d\mathbf{a}$$

using Gauss's law we find

$$\int \mathbf{E} \cdot d\mathbf{a} = \frac{Q}{\varepsilon} \qquad Q = \lambda L$$

thus we find that the current per unit length is given by

$$\boxed{\frac{I}{L} = \frac{\sigma\lambda}{\varepsilon} = \frac{2\pi\sigma}{\ln(r_2/r_1)} V}$$

c) Suppose the battery that maintains the potential difference V is suddenly disconnected from the circuit. Show that charge will leak off the two plates of this capacitor as an exponential function of time. What is the exponential time constant? You should obtain a simple function of ε and σ.

We know that once the battery is disconnected the current will be given as

$$I = -\frac{dQ}{dt} = \frac{\sigma Q}{\varepsilon}$$

and this is a simple integral of the form

$$\int_{Q_i}^{Q_f} \frac{dQ}{Q} = -\int_0^t \frac{\sigma}{\varepsilon} dt$$

which yields

$$Q_f = Q_i e^{-\frac{\sigma}{\varepsilon}t}$$

and the exponential time constant is given by

$$\boxed{k = \frac{\sigma}{\varepsilon}}$$

d) Show that the total energy dissipated by Joule heating as the capacitor discharges completely equals the electrostatic energy that was originally stored in the capacitor.

We know that the work can be expressed as

$$W = \int P dt$$

we know that the power given for Joule heating can be expressed as

$$P = I^2 R$$

thus we find

$$W = \int I^2 R dt$$

but we also know that

$$I = -\frac{dQ}{dt} = -Q_i \frac{\sigma}{\varepsilon} e^{-\frac{\sigma}{\varepsilon}t} \qquad \rightarrow I = Q_i \frac{\sigma}{\varepsilon} e^{-\frac{\sigma}{\varepsilon}t}$$

we also know that

$$R = \frac{V}{I} = \frac{\ln\left(\frac{r_2}{r_1}\right)}{2\pi\sigma L}$$

thus we find that the work is given by

$$W = \left(\frac{\sigma Q_i}{\varepsilon}\right)^2 \ln\left(\frac{r_2}{r_1}\right) \frac{1}{2\pi\sigma L} \int_0^\infty e^{-2\frac{\sigma}{\varepsilon}t} dt = \frac{\varepsilon}{2\sigma} \left(\frac{\sigma Q_i}{\varepsilon}\right)^2 \ln\left(\frac{r_2}{r_1}\right) \frac{1}{2\pi\sigma L}$$

this simplifies into

$$W = \frac{Q^2}{4\pi L\varepsilon} \ln\left(\frac{r_2}{r_1}\right) = \frac{1}{2}\left[\frac{Q}{2\pi L\varepsilon} \ln\left(\frac{r_2}{r_1}\right)\right] Q = \frac{1}{2} QV$$

we know that the work from a capacitor is given by

$$W = \frac{1}{2} CV^2 = \frac{1}{2} QV$$

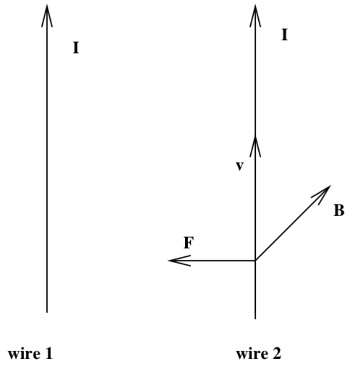
thus we have just shown that these two expressions are equivalent.

Chapter 5

Magnetostatics

5.1 The Lorentz Force Law

The basic problem in classical electrodynamics we have a collection of charges $q_1, q_2, q_3 ...$ which are known as the "source" charges, and we want to know the force they exert on a charge Q. Up till now we were dealing with charges that were at rest *electrostatics* and the time has come to consider the forces between charges *in motion*. Imagine that we have two wires seperated by a few centimeters and when I turn on a current so that it passes up one wire and down the other and the wires jump apart. How do you explain this? Whatever force accounts for the attraction of parallel currents and repulsion of antiparallel ones is not electrostatic in nature. The force responsible for the observed phenomenom is the magnetic force, we can understand this using Figure 1



From Figure 1 we can see that a cross product will allow us to solve for the force. The magnetic force in a charge Q , moving with a velocity **v** in a magnetic field **B**, is

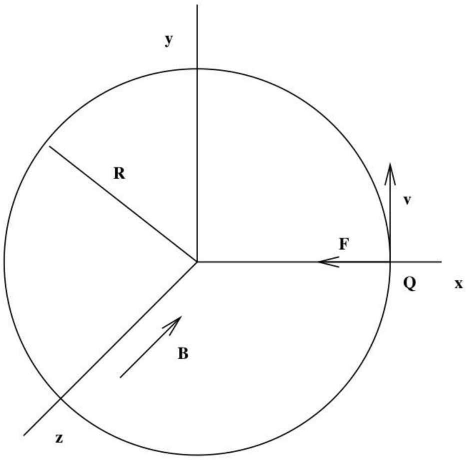
$$\boxed{\mathbf{F}_{\mathrm{mag}} = Q(\mathbf{v} \times \mathbf{B})} \tag{5.1}$$

This is known as the **Lorentz force law**. In the presence of both electric *and* magnetic fields, the net force on Q would be

$$\mathbf{F} = Q[E + (\mathbf{v} \times \mathbf{B})] \tag{5.2}$$

If we take a look at the Lorentz force law we will find some bizarre particle trajectories.

5.1.1 Cyclotron Motion



A charged particle moving about a magnetic field in circular motion, with the magnetic field providing the centripital acceleration is represented in the figure above. A uniform magnetic field points into the page The charge Q is moving counterclockwise, the magnetic force points inward and has a magnitude QvB. We can write

$$\mathbf{F}_{cent} = ma_{cent} = m\frac{v^2}{R}$$

in order to sustain circular motion

$$\mathbf{F}_{cent} = \mathbf{F}_{mag} = QvB = m\frac{v^2}{R}$$

which can also be written

$$\boxed{QvB = m\frac{v^2}{R} \quad \text{or} \quad p = QBR} \tag{5.3}$$

Equation 5.3 is known as the **cyclotron formula** because it describes the motion of a particle in a cyclotron. We have also assumed that the particle moves in a plane perpendicular to **B.** If it starts out with some additional speed $v \parallel parallel$ to **B**, this component of the motion is unaffected by the magnetic field, and the particle moves in a *helix*.

Another feature of the magnetic force law is

$$\boxed{\textbf{Magnetic forces do no work}}$$

For if Q moves an amount $d\mathbf{l} = \mathbf{v}dt$, the work done is

$$dW_{\text{mag}} = \mathbf{F}_{\text{mag}} \cdot d\mathbf{l} = Q(\mathbf{v}\times\mathbf{B})\cdot\mathbf{v}dt = 0$$

this follows because $(\mathbf{v}\times\mathbf{B})$ is perpendicular to $\mathbf{v}$, so $(\mathbf{v}\times\mathbf{B})\cdot\mathbf{v} = 0$.

5.2 Currents

The **current** in a wire is the *charge per unit time* passing a given point. Current is measured in coulombs-per-second, or **amperes** (A)

$$1\ \text{A} = 1\ \text{C/s}$$

A line charge λ traveling down a wire at speed v constitutes a current

$$I = \lambda v$$

Current is actually a vector

$$\mathbf{I} = \lambda \mathbf{v}$$

The magnetic force on a segment of current-carrying wire is given by

$$\mathbf{F}_{mag} = \int (\mathbf{v} \times \mathbf{B}) dq = \int (\mathbf{v} \times \mathbf{B}) \lambda dl = \int (\mathbf{I} \times \mathbf{B}) dl$$

if **I** and $d\mathbf{l}$ both point in the same direction, we can write this as

$$\boxed{\mathbf{F}_{mag} = \int I(d\mathbf{l} \times \mathbf{B}) = I \int (d\mathbf{l} \times \mathbf{B})} \tag{5.4}$$

5.2.1 Surface Currents

When charge flows over a surface, we describe it by the **surface current density, K,** defined by considering a “ribbon” of infinitesimal width $dl \perp$, running parallel to the flow. If the current in this ribbon is $d\mathbf{I}$, the surface current density is

$$\mathbf{K} \equiv \frac{d\mathbf{I}}{dl \perp}$$

this is simply the $current\ per\ unit\ width - perpendicular - to - flow$. This can also be written with a surface charge density

$$\mathbf{K} = \sigma \mathbf{v}$$

and the magnetic force on the surface current is

$$\boxed{\mathbf{F}_{mag} = \int (\mathbf{v} \times \mathbf{B}) \sigma da = \int (\mathbf{K} \times \mathbf{B}) da} \tag{5.5}$$

5.2.2 Volume Currents

When charge flows thru a three-dimensional region, we describe it by the **volume current density, J,** defined by considering a “tube” of infinitesimal cross section $da \perp$, running parallel to the flow. If the current in this ribbon is $d\mathbf{I}$, the volume current density is

$$\mathbf{J} \equiv \frac{d\mathbf{I}}{da \perp} \tag{5.6}$$

this is simply the $current\ per\ unit\ area - perpendicular - to - flow$. This can also be written with a surface charge density

$$\mathbf{J} = \rho \mathbf{v}$$

and the magnetic force on the surface current is

$$\boxed{\mathbf{F}_{mag} = \int (\mathbf{v} \times \mathbf{B}) \rho d\tau = \int (\mathbf{J} \times \mathbf{B}) d\tau} \tag{5.7}$$

Equation 5 can be written as

$$I = \int_S J da \perp = \int_S \mathbf{J} \cdot d\mathbf{a}$$

which describes the current crossing a surface S. The dot product serves neatly to pick out the approriate component of $d\mathbf{a}$. In particular, the total charge per unit time leaving a volume V is

$$\oint_S \mathbf{J}\cdot d\mathbf{a} = \int_V (\nabla\cdot\mathbf{J})d\tau$$

because charge is conserved, whatever flows out thru a surface must come at the expense of that remaining inside, this can be mathematically stated as

$$\nabla\cdot\mathbf{J} = -\frac{\partial\rho}{\partial t} \tag{5.8}$$

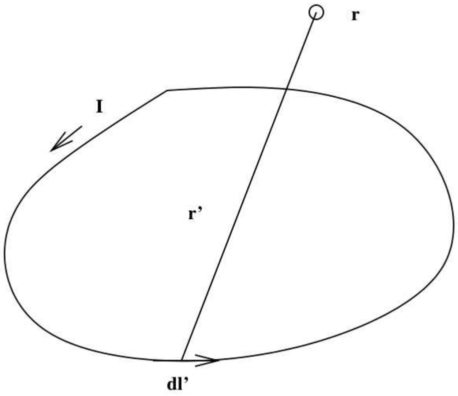
this is known as the **continuity equation**. For future reference, lets summarize what we know

$$\sum_{i=1}^{n}(\ )q_i\mathbf{v}_i \sim \int_{line}(\ )\mathbf{I}dl \sim \int_{surface}(\ )\mathbf{K}da \sim \int_{volume}(\ )\mathbf{J}d\tau$$

this corresponds to $q\sim\lambda dl\sim\sigma da\sim\rho d\tau$ for the various charge distributions.

5.3 The Biot-Savart Law

5.3.1 The Magnetic Field of a Steady Current



The magnetic field of a steady current ius given by the **Biot-Savart Law**

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{I}\times\hat{\mathbf{r}}'}{r'^2} = \frac{\mu_0}{4\pi}I\int\frac{d\mathbf{l}'\times\hat{\mathbf{r}}'}{r'^2}$$

The integration is along the current path, in the direction of the flow; $d\mathbf{l}'$ is an element of length along th wire, and $\mathbf{r}'$, as always, is the vector from the source to the point $\mathbf{r}$. The constant μ_0 is called the **permeability of free space**.

$$\mu_0 = 4\pi\times10^{-7}\mathrm{N/A}^2$$

These units are such that **B** itself comes out in newtons per ampere-meter (as required by the Lorentz force law) or **teslas** (T)

$$1\ \mathrm{T} = 1\ \mathrm{N/(A\cdot m)}$$

The Biot-Savart law plays a role analogous to Coulomb's law in electrostatics. The inverse square law is common in both laws.

5.3.1.1 Wire segment

The magnetic field a distance s from a long straight wire carrying a steady current I is given by

$$B = \frac{\mu_0 I}{4\pi s}(\sin\theta_2 - \sin\theta_1)$$

this is the general solution for any line segment, thus the magnetic field of an infinitely long wire would be given as.

$$B = \frac{\mu_0 I}{2\pi s} \tag{5.9}$$

because $\theta_2 = -\theta_1 = \pi/2$.

5.3.1.2 Circular loop of wire

The magnetic fielda distance z above the center of a circular loop of wire with radius R, which carries a steady current I is given as

$$B(z) = \frac{\mu_0 I}{2}\frac{R^2}{(R^2+z^2)^{3/2}}$$

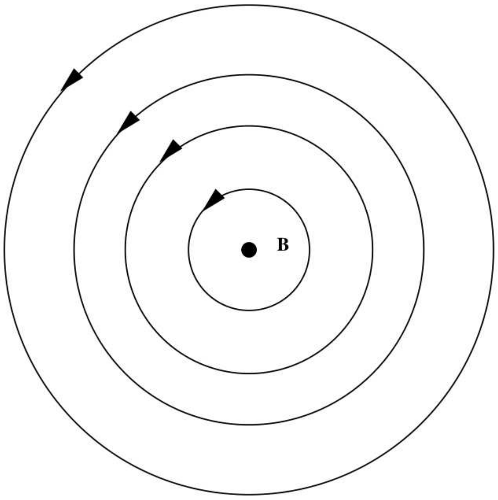
For surface and volume currents the Biot-Savart law becomes

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{K}(\mathbf{r}'') \times \hat{\mathbf{r}}'}{r'^2} da'' \quad \text{and} \quad \mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}'') \times \hat{\mathbf{r}}'}{r'^2} d\tau''$$

5.4 The Divergence and Curl of B

5.4.1 Straight-Line Currents

The magnetic field of an infinitely straight wire is shown as



According to Equation 7, the integral of **B** around a circular path of radius s, centered at the wire is

$$\oint \mathbf{B}\cdot d\mathbf{l} = \oint \frac{\mu_0 I}{2\pi s} dl = \frac{\mu_0 I}{2\pi s}\oint dl = \mu_0 I$$

Notice that the solution is independent of s, thats because B decreases at the same rate that the circumference increases. Now suppose we have a bundle of straight wires. Each wire that passes thru our loop contributes $\mu_0 I$, and those outside contribute nothing. The line integral will then be

$$\int \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}} \tag{5.10}$$

where I_{enc} stands for the total current enclosed by the integration path. If the flow of charge is represented by a volume current density $\mathbf{J}$, the enclosed current is

$$I_{\text{enc}} = \int \mathbf{J} \cdot d\mathbf{a}$$

with the integral taken over the surface bounded by the loop. Applying Stoke's theorem to Equation 5.8 gives

$$\int (\nabla \times \mathbf{B}) \cdot d\mathbf{a} = \mu_0 \int \mathbf{J} \cdot d\mathbf{a}$$

and hence

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{5.11}$$

We also know that the divergence of $\mathbf{B}$ is given by

$$\nabla \cdot \mathbf{B} = 0$$

the divergence of the magnetic field is 0.

5.4.2 Applications of Ampere's Law

The equation for the curl of $\mathbf{B}$

$$\boxed{\nabla \times \mathbf{B} = \mu_0 \mathbf{J}}$$

is called **Ampere's law** (in differential form). It can be converted to integral form by the usual device of applying one of the fundemental theorems-in this case Stoke's theorem

$$\int (\nabla \times \mathbf{B}) \cdot d\mathbf{a} = \oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int \mathbf{J} \cdot d\mathbf{a}$$

Now, $\int \mathbf{J} \cdot d\mathbf{a}$ is the total current passing through the surface, which we call I_{enc} , thus

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}$$

This is the integral version of Ampere's law; it is generalized to arbitrary steady currents. Just as the Biot-Savart law plays a role in magnetostatics that Coulomb's law assumed in electrostatics, so Ampere's plays the role of Gauss's:

Electrostatics:	Coulomb	→	Gauss
Magnetostatics:	Biot-Savart	→	Ampere

In particular, for currents with appropriate symmetry, Ampere's law in integral form offeres a lovely and extraordinarily efficient means for calculating the magnetic field.

5.4.3 Comparison of Magnetostatics and Electrostatics

The divergence and curl of the electrostatic field are

$$\begin{aligned}\nabla \cdot \mathbf{E} &= \frac{\rho}{\varepsilon_0}, \quad \text{Gauss's law}\\ \nabla \times \mathbf{E} &= 0 \quad \text{no name}\end{aligned}$$

These are **Maxwell's equations** for electrostatics. The divergence and curl of the magnetostatic field are

$$\begin{aligned}\nabla \cdot \mathbf{B} &= 0 \quad \text{no name}\\ \nabla \times \mathbf{B} &= \mu_0 I_{\text{enc}} \quad \text{Ampere's law}\end{aligned}$$

These are Maxwell's equations for magnetostatics. Maxwell's equations and the force law

$$\mathbf{F} = Q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

constitute the most elegant formulation of electrostatics and magnetostatics.

5.5 Magnetic Vector Potential

5.5.1 The Vector Potential

Just as $\nabla \times \mathbf{E} = 0$ permitted us to introduce a scalar potential (V) in electrostatics,

$$\mathbf{E} = -\nabla V$$

so $\nabla \cdot \mathbf{B} = 0$ invites the introduction of a *vector* potential $\mathbf{A}$ in magnetostatics

$$\mathbf{B} = \nabla \times \mathbf{A}$$

The potential formulation automatically takes care of $\nabla \cdot \mathbf{B} = 0$ (since the divergence of a curl is always zero); there remains Ampere's law

$$\nabla \times \mathbf{B} = \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}$$

if we let

$$\nabla \cdot \mathbf{A} = 0$$

with this condition to $\mathbf{A}$, Ampere's law becomes

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$$

This is nothing but Poisson's equation. This is actually *three* Poisson's equations, on for each cartesian component. Assuming $\mathbf{J}$ goes to zero at infinity, we can read off the solution

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}'')}{r'} d\tau''$$

5.5.2 Summary; Magnetostatic Boundary Conditions

Just as the electric field suffers a discontinuity at a surface *charge*, so the magnetic field is discontinous at a surface *current*. Only this time it is the *tangential* that changes. We can write this as

$$\mathbf{B}_{\text{above}} - \mathbf{B}_{\text{below}} = \mu_0(\mathbf{K}\times\hat{\mathbf{n}})$$

where $\hat{\mathbf{n}}$ is the unit vector perpendicular to the surface, pointing "outward". Like the scalar potential in electrostatics, the vector potential is continuous across any boundary

$$\mathbf{A}_{\text{above}} = \mathbf{A}_{\text{below}}$$

for $\nabla\cdot\mathbf{A} = 0$ guarantees that the n ormal component is continuous, and $\nabla\times\mathbf{A} = \mathbf{B}$, in the form

$$\oint \mathbf{A}\cdot d\mathbf{l} = \int \mathbf{B}\cdot d\mathbf{a} = \Phi$$

means that the tangential components are continuous. But the derivative of **A** inherits the discontinuity of **B**.

$$\frac{\partial\mathbf{A}_{\text{above}}}{\partial n} - \frac{\partial\mathbf{A}_{\text{below}}}{\partial n} = -\mu_0\mathbf{K}$$

5.6 Multipole Expansion of the Vector Potential

If you want an approximation formula for the vector potential of a localized current distribution, valid at distant points, a multipole expansion is in order. The vector potential of a current loop can be written as

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\oint (r')^n P_n(\cos\theta')d\mathbf{l}'$$

or more explicetly

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\left[\frac{1}{r}\oint d\mathbf{l}' + \frac{1}{r^2}\oint r'\cos\theta' d\mathbf{l}' + \frac{1}{r^3}\oint (r')^2\left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right)d\mathbf{l}' + \ldots\right]$$

as in the multipole expansion of V, we call the first term (which goes like $1/r$) the **monopole** term, the second (which goes like $1/r^2$) **dipole**, the third **quadrupole**, and so on. It happens that the magnetic monopole term is always zeror, for the integral is just the total vector displacement around a closed loop

$$\oint d\mathbf{l}' = 0$$

this reflects that there are no magnetic monopoles in nature. In the absence of any monopole contribution, the dominent term is the dipole (except in the rare case where it too, vanishes)

$$\mathbf{A}_{\text{dip}} = \frac{\mu_0 I}{4\pi r^2}\oint r'\cos\theta' d\mathbf{l}' = \frac{\mu_0 I}{4\pi r^2}\oint(\hat{\mathbf{r}}\cdot\mathbf{r}')d\mathbf{l}'$$

this integral can be written in a more illuminating way if we invoke Eq 1.108 from Griffith's, with $\mathbf{c} = \hat{\mathbf{r}}$:

$$\oint(\hat{\mathbf{r}}\cdot\mathbf{r}')d\mathbf{l}' = -\hat{\mathbf{r}}\times\int d\mathbf{a}'$$

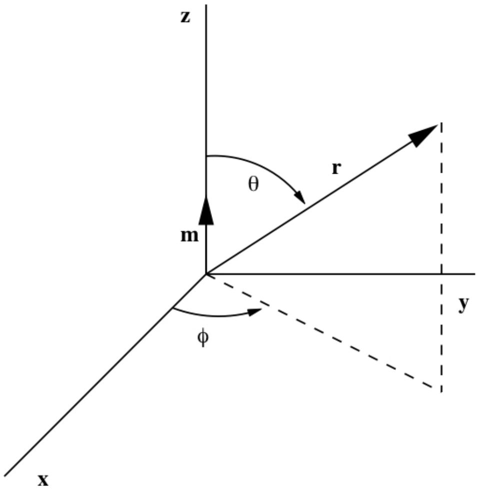
then

$$\mathbf{A}_{\text{dip}}\left(\mathbf{r}\right)=\frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2} \tag{5.12}$$

where **m** is the **magnetic dipole moment:**

$$\mathbf{m}\equiv I\int d\mathbf{a}=I\mathbf{a}$$

where **a** is the area vector. The magnetic field of a pure dipole is easiest to calculate if we put **m** at the origin and let it point in the $z-$direction



According to equation 5.10, the potential at point (r,θ,ϕ) is

$$\mathbf{A}_{\text{dip}}(\mathbf{r})=\frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{\phi}$$

and hence

$$\mathbf{B}_{\text{dip}}(\mathbf{r})=\nabla\times\mathbf{A}=\frac{\mu_0 m}{4\pi r^3}(2\cos\theta\hat{\mathbf{r}}+\sin\theta\hat{\phi})$$

5.7 Problems with Solutions

Problem # 1 Polygonal electromagnets

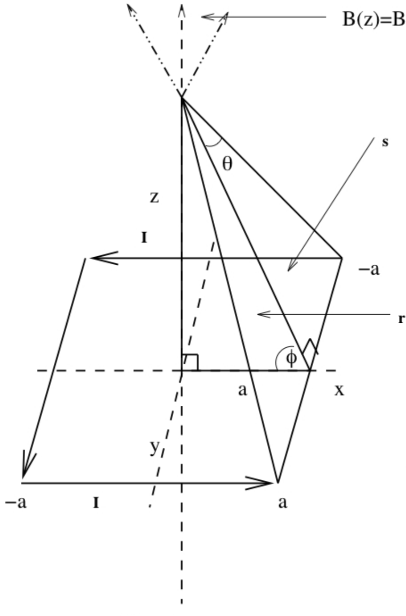


Figure 1

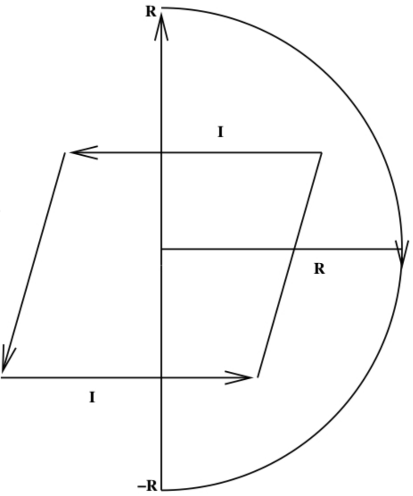


Figure 2

a) A square loop of wire, with size $2a \times 2a$ lies in the xy-plane centered at the origin. Calculate the field everywhere on the z -axis produced by a current I through this loop.

We know that a line segment produces a magnetic field a distance s away of

$$B(s) = N\frac{\mu_0 I}{4\pi s}2\sin\theta = N\frac{\mu_0 I}{2\pi s}\sin\theta$$

where N represents the number of segments, in our case this is just 4, so we find

$$B(s) = \frac{2\mu_0 I}{\pi s}\sin\theta$$

we can see from the diagram that

$$\sin\theta = \frac{a}{r} = \frac{a}{\sqrt{a^2+s^2}} = \frac{a}{\sqrt{2a^2+z^2}} \qquad s = \sqrt{a^2+z^2}$$

thus we find

$$B(\mathbf{r}) = \frac{2\mu_0 I}{\pi s}\frac{a}{\sqrt{2a^2+z^2}}$$

this gives us our magnetic field as a function of $\mathbf{r}$, We know that the horizontal components will cancel by symmetry and we will be left with a vertical component in the $\hat{\mathbf{z}}$ direction, i.e

$$B_z = B\cos\phi = B\frac{a}{s} = \boxed{\frac{2\mu_0 I}{\pi}\frac{a^2}{(a^2+z^2)\sqrt{2a^2+z^2}}}$$

given that

$$\cos\phi = \frac{a}{s}$$

b) What is the line integral of the field along the z -axis from $z = \infty$ to $z = -\infty$? Could you have expected this result based on Ampere's law? Explain.

We must now find out what the integral yields

$$\int_{-\infty}^{\infty} B_z dz = \frac{2\mu_0 I a^2}{\pi} \int_{-\infty}^{\infty} \frac{dz}{(a^2+z^2)(2a^2+z^2)^{1/2}}$$

we can make a substitution of the form

$$z = \sqrt{2}a\tan\theta \quad dz = \sqrt{2}a\sec^2\theta d\theta$$

plugging this into the integral gives

$$\frac{2\mu_0 I a^2}{\pi}\int_{-\pi/2}^{\pi/2} \frac{\sqrt{2}a\sec^2\theta d\theta}{a^2(1+2\tan^2\theta)(2a^2(1+\tan^2\theta))^{1/2}} = \frac{2\mu_0 I a^2}{\pi}\int_{-\pi/2}^{\pi/2}\frac{\sec\theta d\theta}{a^2(1+2\tan^2\theta)}$$

we can now multiply both top and bottom by $\cos^2\theta$ to give

$$\frac{2\mu_0 I a^2}{\pi}\int_{-\pi/2}^{\pi/2}\frac{\sec\theta d\theta}{a^2(1+2\tan^2\theta)}\frac{\cos^2\theta}{\cos^2\theta} = \frac{2\mu_0 I a^2}{\pi}\int_{-\pi/2}^{\pi/2}\frac{\cos\theta d\theta}{a^2(2\sin^2\theta+\cos^2\theta)}$$

knowing

$$\cos^2\theta = 1-\sin^2\theta$$

allows us to simplify to

$$\frac{2\mu_0 I a^2}{\pi}\int_{-\pi/2}^{\pi/2}\frac{\cos\theta d\theta}{a^2(1+\sin^2\theta)}$$

doing another substitution

$$k = \sin\theta \qquad dk = \cos\theta d\theta$$

which gives us

$$\frac{2\mu_0 I a^2}{\pi}\int_{-1}^{1}\frac{dk}{a^2(1+k^2)} = \left[\frac{2\mu_0 I}{\pi}\tan^{-1}(k)\right]_{-1}^{1} = \mu_0 I$$

thus

$$\frac{2\mu_0 I a^2}{\pi}\int_{-\infty}^{\infty}\frac{dz}{(a^2+z^2)(2a^2+z^2)^{1/2}} = \boxed{\mu_0 I}$$

This is the line integral from $-R$ to R where $R \to \infty$, what about the line integral of the semi-circle? We can solve this integral by understanding how the magnetic field behaves far way from the z axis, we know from the Biot-Savart law that the magnetic field goes like

$$\mathbf{B} \propto \frac{1}{R^2}$$

If we let $R \to \infty$ we know that the integral along the semi-circle will go to zero, this exactly what we would expect from Ampere's law, which just says that if you enclose a current with an infinite loop you should get the total current enclosed.

c)Now find the magnetic field at the center of a n-sided regular polygon, letting the perpendicular distance from the center to any side be a. Show that the result approaches that for a circular loop of radius a in the limit $n \to \infty$.

We know that the solution for the magnetic field of a line segment on a plane is given by

$$B(s) = \frac{\mu_0 I \sin\theta}{2\pi s}$$

now we need to find a relationship between the angle θ and the number of sides of a polygon.

$$\begin{aligned}
\text{4 sides} \quad 4(2\theta) &= 2\pi \ \rightarrow \theta = \frac{\pi}{4} \\
\text{5 sides} \quad 5(2\theta) &= 2\pi \ \rightarrow \theta = \frac{\pi}{5} \\
\text{6 sides} \quad 6(2\theta) &= 2\pi \ \rightarrow \theta = \frac{\pi}{6} \\
n \text{ sides} \quad n(2\theta) &= 2\pi \ \rightarrow \theta = \frac{\pi}{n}
\end{aligned}$$

using this defenition for θ in the expression for the magnetic field yields

$$B(s) = \frac{\mu_0 I \sin(\pi/n)}{2\pi s}$$

we also know that

$$\lim_{n\to\infty} \sin(\pi/n) = \frac{\pi}{n}$$

thus we find

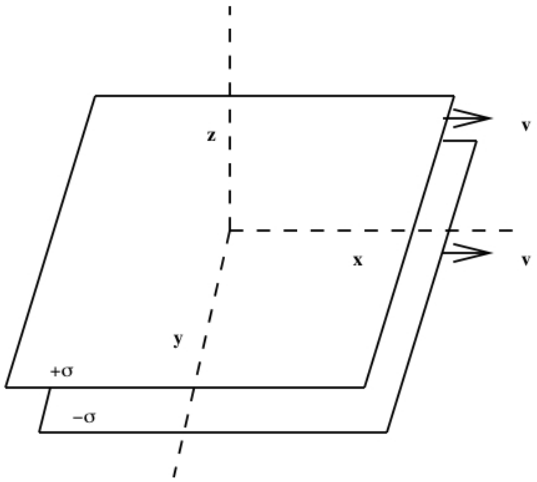
$$B(s) = \frac{\mu_0 I}{2ns}$$

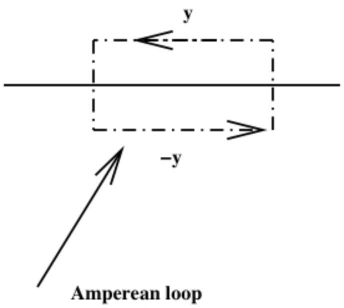
this is only for 1 side, for n sides we find

$$B(s) = n\left(\frac{\mu_0 I}{2ns}\right) = \boxed{\frac{\mu_0 I}{2s}}$$

which is the magnetic field of a loop of current with radius s.

Problem # 2





A large parallel-plate capacitor with uniform surface charge σ on the upper plate and $-\sigma$ on the lower is moving with a constant speed v, as shown in Figure 3.

a) Find the magnetic field bewteen the plates and also above and below them.

We know from Ampere's law

$$\oint \mathbf{B} \cdot d\mathbf{a} = \mu_0 I_{enc}$$

where

$$I_{enc} = \sigma v L \qquad d\mathbf{a} = 2L\hat{\mathbf{z}}$$

plugging this into Ampere's law gives

$$\mathbf{B}2L = \mu_0 \sigma v L$$

thus the magnetic field is given by

$$\mathbf{B} = \frac{\mu_0 \sigma v}{2}$$

thus

$$\begin{aligned} B &= -\frac{\mu_0 \sigma v}{2}\hat{\mathbf{y}} \quad \text{above top plate} \\ B &= +\frac{\mu_0 \sigma v}{2}\hat{\mathbf{y}} \quad \text{below top plate} \end{aligned}$$

and the bottom plate is analogous to the top plate except all the signs change due to the $-\sigma$ surface charge

$$\begin{aligned} B &= +\frac{\mu_0 \sigma v}{2}\hat{\mathbf{y}} \quad \text{above bottom plate} \\ B &= -\frac{\mu_0 \sigma v}{2}\hat{\mathbf{y}} \quad \text{below bottom plate} \end{aligned}$$

combining our results we find

$$\begin{aligned} B &= 0 \quad \text{above and below the plates} \\ B &= \mu_0 \sigma v \hat{\mathbf{y}} \quad \text{in between the plates} \end{aligned}$$

b) Find the magnetic force per unit area on the upper plate, including its direction.

We know that

$$F_{mag} = q(\mathbf{v} \times \mathbf{B}) = \int \sigma(\mathbf{v} \times \mathbf{B}) da = \int (\mathbf{K} \times \mathbf{B}) da$$

where

$$\mathbf{K}_t = \sigma \mathbf{v} \hat{\mathbf{x}}$$

and we also know that the force is caused by the bottom plate, thus

$$\mathbf{B} = \frac{\mu_0 \sigma v}{2}\hat{\mathbf{y}}$$

so we find that the cross product yields

$$\mathbf{K} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \sigma \mathbf{v} & 0 & 0 \\ 0 & \frac{\mu_0 \sigma \mathbf{v}}{2} & 0 \end{vmatrix} = \frac{\mu_0 \sigma^2 v^2}{2}\hat{\mathbf{z}}$$

thus the force per unit area is given by

$$F_{mag} = \int (\mathbf{K}\times\mathbf{B})da = \frac{\mu_0\sigma^2 v^2}{2}\hat{\mathbf{z}}A$$

$$\boxed{\frac{F_{mag}}{A} = \frac{\mu_0\sigma^2 v^2}{2}\hat{\mathbf{z}}}$$

c) At what speed v would the magnetic force balance the electrical force?

We know that

$$F_e = F_{mag}$$

but we also know

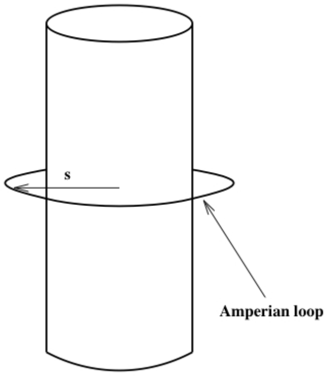
$$F_e = q\frac{\sigma}{2\varepsilon_0} \qquad F_{mag} = q\frac{\mu_0\sigma v^2}{2}$$

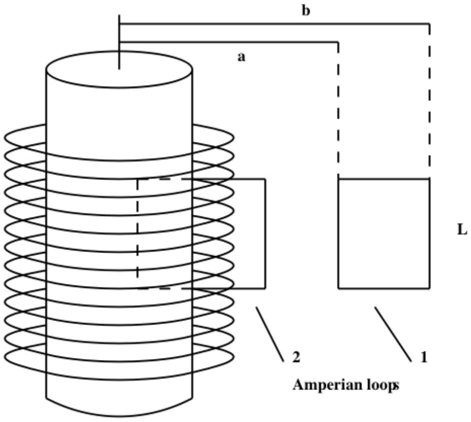
thus

$$v^2 = \frac{1}{\varepsilon_0\mu_0} \quad \rightarrow \quad v = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = \boxed{c}$$

thus we have shown that these two forces would be equal at a velocity of the speed of light.

Problem # 3





Suppose a very long solenoid carries a current of 10 A and has 1000 turns per cm. Find the radial force per unit length f on one turn of the winding. Careful: the magnetic field for an infinite solenoid is discontinuous - how did we handle such discontinuities to determine the pressure on the surface of a charged conductor? Such forces pose severe design constraints on the construction of large, typically superconducting, electromagnets.

We know that the force per unit length is given as

$$f = \frac{F_{mag}}{L}$$

and we can find F_{mag} using

$$F_{mag} = \int (\mathbf{v}\times\mathbf{B})dq = \int (\mathbf{v}\times\mathbf{B})\lambda dl = (\sigma\mathbf{v}\times\mathbf{B})da = (\mathbf{K}\times\mathbf{B})da = I\int (d\mathbf{l}\times\mathbf{B})$$

and since we know that **K** and **B** are orthogonal we find that the force per unit length is

$$\frac{F_{mag}}{L} = I\mathbf{B}$$

we can find the magnetic field by using Ampere's law, we also know that the magnetic field runs parallel to the axis and by the righthand rule we know that it points upward inside the solenoid and downward outside. For the loop inside the solenoid (loop 2) we find

$$\oint \mathbf{B} \cdot d\mathbf{l} = BL = \mu_0 n I_{enc} = \mu_0 n I L$$

for the loop outside (loop 1) we find

$$\oint \mathbf{B} \cdot d\mathbf{l} = BL = \mu_0 n I_{enc} = 0$$

due to the fact that the loop does not enclose any current. Thus we find that the magnetic field for this solenoid is given by

$$\mathbf{B} = \mu_0 n I \hat{\mathbf{z}} \text{ inside} \quad \mathbf{B} = 0 \quad \text{outside}$$

we also know that the magnetic field of an infinite solenoid is discontinous at a surface current, which is the same case as the discontinuity of the electric field at a surface charge. This means that the magnetic field is just the average of the magnetic field below and and the magnetic field above,

$$\mathbf{B} = \frac{\mathbf{B}_{above} + \mathbf{B}_{below}}{2} = \frac{\mu_0 n I}{2}\hat{\mathbf{z}}$$

thus we find that the force per unit length is

$$\frac{F_{mag}}{L} = \frac{\mu_0 n I^2}{2}\hat{\mathbf{z}}$$

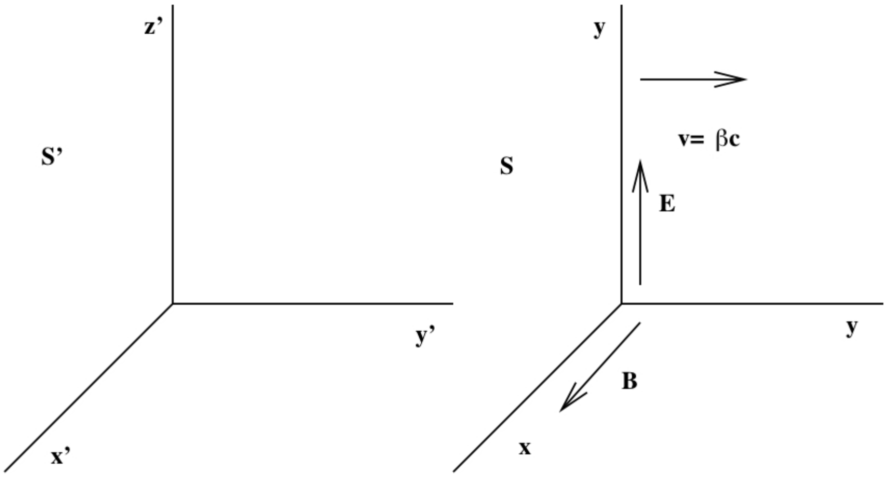
plugging in the constants yield

$$\frac{F_{mag}}{L} = \boxed{6.28\frac{\text{N}}{\text{m}}}$$

Problem # 4

Cycloid motion

Griffiths' Ex. 5.2 obtains a cycloid solution for motion of a charged particle in crossed electric and magnetic fields using the Lorentz force law in the laboratory frame. Read this example, and then solve the same problem using an alternate approach:



a) From relativity, but in the limit of velocities small compared to the speed of light, if frame S moves with velocity $\vec{\beta}c$ with respect to (laboratory) frame S' , the electric field is transformed as

$$\mathbf{E}' = \mathbf{E} + \vec{\beta} \times c\mathbf{B}$$

Taking $\mathbf{B}$ along $\hat{\mathbf{x}}$ and $\mathbf{E}$ along $\hat{\mathbf{z}}$, as in the text, and assuming $E \ll cB$ in magnitude, choose the simplest $\vec{\beta}$ so that $\mathbf{E}'$ vanishes.

We know that for $\mathbf{E}'$ to vanish, $\vec{\beta} \times c\mathbf{B} = -\mathbf{E}$, thus

$$\vec{\beta} \times c\mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \beta_x & \beta_y & \beta_z \\ cB & 0 & 0 \end{vmatrix} = -\mathbf{E}\hat{\mathbf{z}}$$

which gives

$$-cB\beta_y\hat{\mathbf{z}} + cB\beta_z\hat{\mathbf{y}} = -E\hat{\mathbf{z}}$$

so we find

$$\boxed{\beta = \frac{E}{cB}\hat{\mathbf{y}}}$$

b) The two frames coincide at $t = 0$. Using the Galilei transformation for velocities (valid for $\beta \ll 1$), if the particle is at rest at $t = 0$ in frame S , what is its initial velocity in frame S' ?

we know that the Galilei transformations for velocities can be expressed as

$$\mathbf{u}' = \mathbf{u} - \mathbf{v}$$

or

$$\begin{aligned} u'_x &= u_x - \beta_x c \\ u'_y &= u_y - \beta_y c \\ u'_z &= u_z - \beta_z c \end{aligned}$$

but we know that

$$\beta_x = \beta_z = 0$$

thus we find that the initial velocity in the S' frame is

$$u'{}_y = u_y - \beta_y c \qquad u_y = 0$$

so

$$\boxed{u'_y = -\beta_y c}$$

c) The magnetic field transforms in this case as

$$c\mathbf{B}' = c\mathbf{B} - \vec{\beta} \times \mathbf{E}$$

Because $E \ll cB$ and $\beta \ll 1$, approximate **B’** to be equal to $\mathbf{B}$. Now, in frame S' solve for the motion of the particle. Using the Galilei transformation for positions, determine the motion of the particle in the lab frame.

If we assume that $\mathbf{B}' = \mathbf{B}$ than

$$\beta \times \mathbf{E} = 0 \qquad \mathbf{E} = 0$$

so we can see that

$$\mathbf{F}_{mag} = q(\mathbf{v}' \times \mathbf{B}')$$

and lets assume that$\mathbf{v}'$ has velocity components

$$\mathbf{v}' = \langle \dot{x}, \dot{y}, \dot{z} \rangle \qquad \mathbf{B}' = \langle B_x, 0, 0 \rangle$$

thus we find

$$\mathbf{v}' \times \mathbf{B}' = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \dot{x} & \dot{y} & \dot{z} \\ B & 0 & 0 \end{vmatrix} = B\dot{z}\hat{\mathbf{y}} - B\dot{y}\hat{\mathbf{z}}$$

the magnetic force is now given as

$$\mathbf{F}_{mag} = q(B\dot{z}\hat{\mathbf{y}} - B\dot{y}\hat{\mathbf{z}}) = m\vec{a} = m(\ddot{y}\hat{\mathbf{y}} + \ddot{z}\hat{\mathbf{z}})$$

which reduces to

$$qB\dot{z} = m\ddot{y} \qquad -qB\dot{y} = m\ddot{z}$$

we can define the cylotron frequency as

$$\omega = \frac{qB}{m}$$

and we find

$$\ddot{y} = \omega\dot{z} \qquad \ddot{z} = -\omega\dot{y}$$

if we differentiate the first to get

$$\dddot{y} = \omega\ddot{z} = \omega^2\dot{y}$$

which becomes

$$\dddot{y} - \omega^2\dot{y} = 0$$

this has a general solution of the form

$$\begin{aligned} y'(t) &= A_1 \cos\omega t + A_2 \sin\omega t + A_3 \\ \dot{y} &= -A_1\omega \sin\omega t + A_2\omega \cos\omega t \\ \ddot{y} &= -A_1\omega^2 \cos\omega t - A_2\omega^2 \sin\omega t \\ \dddot{y} &= A_1\omega^3 \sin\omega t - A_2\omega^3 \cos\omega t \end{aligned}$$

and we can see that this solution works for our differential equation. Now to find $z'(t)$ we can use the first derivative of y' which yields\

$$\begin{aligned} \ddot{z} &= -\omega y = A_1\omega^2 \sin\omega \dot{t} - A_2\omega^2 cos\omega t \\ \dot{z} &= -A_1\omega \cos\omega t - A_2\omega \sin\omega t \\ z'(t) &= -A_1 \sin\omega t + A_2 \cos\omega t + A_4 \end{aligned}$$

now we have two expressions for the motion of the particle,

$$\begin{aligned} y'(t) &= A_1 \cos\omega t + A_2 \sin\omega t + A_3 \\ z'(t) &= -A_1 \sin\omega t + A_2 \cos\omega t + A_4 \end{aligned}$$

now we need to find what these constants are using our boundary coditions

$$\begin{aligned} y'(0) &= z'(0)=0 \quad \text{boundary condtion \#1} \\ v'_y(0) &= -c\beta\hat{\mathbf{y}} \quad v'_z(0)=0 \quad \text{boundary condtion \#2} \end{aligned}$$

from these two boundary conditions we find

$$\begin{aligned} 0 &= A_2+A_4 \quad A_2=-A_4 \quad \text{from 1} \\ A_2 &= -\frac{c\beta}{\omega} \quad A_1=A_3=0 \quad A_4=\frac{c\beta}{\omega} \end{aligned}$$

thus we find that the solutions that describe this motion are given by

$$\boxed{y'(t)=-\frac{c\beta}{\omega}\sin\omega t} \qquad \boxed{z'(t)=-\frac{c\beta}{\omega}\cos\omega t+\frac{c\beta}{\omega}}$$

Problem # 5

6) Along with the necessary addition made by Maxwell, Ampere's law reads

$$\nabla\times\mathbf{B}=\mu_0\mathbf{J}+\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}$$

Taking the divergence of this equation, show that electric charge is conserved globally. We thus see that the conservation of charge is tied fundamentally with the validity of Maxwell's equations.

taking the divergence of both sides of this equation yields

$$\nabla\cdot(\nabla\times\mathbf{B})=\mu_0\nabla\cdot\mathbf{J}+\frac{1}{c^2}\frac{\partial}{\partial t}(\nabla\cdot\mathbf{E})$$

but we know that

$$\nabla\cdot(\nabla\times\mathbf{B})=0 \qquad \nabla\cdot\mathbf{E}=\frac{\rho}{\varepsilon_0}$$

this gives us

$$\mu_0\nabla\cdot\mathbf{J}+\frac{1}{c^2\varepsilon_0}\frac{\partial\rho}{\partial t}=0$$

but

$$c^2=\frac{1}{\mu_0\varepsilon_0}$$

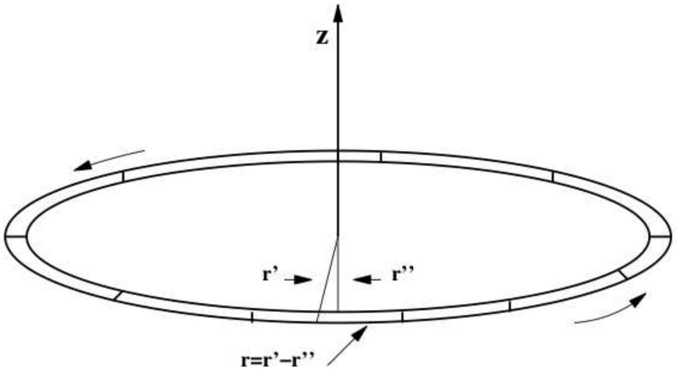
and so we find

$$\nabla\cdot\mathbf{J}=-\frac{\partial\rho}{\partial t}$$

because the charge is conserved, whatever flows out through a surface must come at the expense of whatever is inside. Therefore this is the mathematical statement of local charge conservation. This is generally refered to the continuity equation.

Problem # 6 On the gyromagnetic ratio

A thin uniform donut, carrying a charge Q and mass M, rotates about its axis as shown in Figure 1.



a). Find the ratio of its magnetic dipole moment to its angular momentum. This is called the **gyromagnetic ratio** (or **magnetomechanical ratio**).

We know that the magnetic moment is defined as

$$\mathbf{m} = \gamma \mathbf{L}$$

where γ is the gyromagnetic ratio and $\mathbf{L}$ is the angular momentum of the system. We also know that the magnetic moment can be defined as

$$\mathbf{m} = Ia = I\pi r^2$$

where I is the current and a is the area. The angular momentum of a hoop is given by

$$\mathbf{L} = \mathbf{I}\omega$$

where $\mathbf{I}$ is now the moment of inertia of a hoop, which is given by $\mathbf{I} = mr^2$, thus we find the angular momentum to be given as

$$\mathbf{L} = mr^2\omega$$

we also know that the current can be expressed as

$$I = \frac{Q}{T} \quad T = \frac{2\pi}{\omega} \quad I = \frac{Q\omega}{2\pi}$$

putting all this together we find that the gyromagnetic ratio is given by

$$\gamma = \frac{\mathbf{m}}{\mathbf{L}} = \frac{Q\omega\pi r^2}{2\pi m r^2 \omega} = \boxed{\frac{Q}{2m}}$$

b). What is the gyromagnetic ratio for a uniform spinning sphere? [This requires no new calculation; simply decompose the sphere into infinitesimal rings, and apply the result form part a).]

From part a) we can see that the gyromagnetic ratio does not depend on the radius and thus decomposing the sphere into infinitesimal rings will still yield the same ratio. The gyrommagnetic ratio is only dependent on the total charge Q of the system and the mass M. This can be expressed mathematically as

$$\frac{\sum \mathbf{m}}{\sum \mathbf{L}} = \frac{\sum \frac{q_i}{2} r_i^2 \omega}{\sum m_i r_i^2 \omega} = \frac{Q}{2m}$$

c). According to quantum mechanics, the angular momentum of a spinning electron is $\frac{1}{2}\hbar$, where $\hbar$ is Planck's constant. What, then is the electron's magnetic dipole moment, in A$\cdot m^2$? [This semiclassical value is actually of by a factor of almost exactly 2. Dirac's relativistic electron theory got the 2 right, and Feynman, Schwinger, and Tomonaga later calculated tiny further corrections. The determination of the electron's magnetic dipole moment remains the finest achievment of quantum electrodynamics, and exhibits perhaps the most stunningly precise agreement between theory and experiment in all of physics. Incidentally, the quantity $(e\hbar/2m)$, where e is the charge of the electron and m is its mass, is called the **Bohr magneton**.]

since we know that the magnetic moment is defined as

$$\mathbf{m} = \gamma \mathbf{L}$$

and the angular momentum of a spinning electron is given by

$$\mathbf{L} = \frac{1}{2}\hbar$$

we find the magnetic moment of an electron to be given as

$$\mathbf{m} = \frac{Q}{4m}\hbar = \boxed{4.61 \times 10^{-24} \text{A m}^2}$$

Chapter 6

Magnetic Fields in Matter

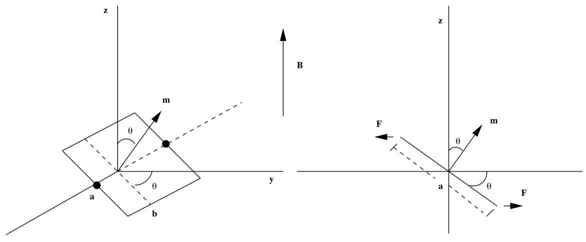
6.1 Magnetization

All magnetic phenomenom are due to electric charges in motion, if you could examine a piece of magnetic material on an atomic scale you would find tiny currents: electrons orbiting around the nucleus and electrons spinning about their axes. For macroscopic purposes, these current loops are so small that we may treat them as magnetic dipoles. Ordinarily, they cancel each other out because of the random orientation of the atoms. But when a magnetic field is applied, a net alignment of these magnetic dipoles occur, and the medium becomes magnetically polarized, or **magnetized**.

Unlike electric polarization, which is almost always in the direction of the field, some materials acquire a magnetization *parallel* to **B** (paramagnets) and some *opposite* to **B** (diamagnets). A few substances called *ferromagnets*, retain their magnetization even after the external field has been removed.

6.2 Torques and Forces on Magnetic Dipoles

A magnetic dipole experiences a torque in a magnetic field just as a electric dipole does in an electric field. Lets calculate the torque on a rectangular current loop in a uniform magnetic field **B**.



We have centered the loop at the origin, and a tilt of angle θ from the z axis toward the y axis and the magnetic field **B** points in the z dircetion. The forces on the two sloping sides cancel(they tend to stretch the loop but not rotate it). The forces on the "horizontal" sides are likewise equal and opposite(so the net force on the loop is zero), but they do generate a torque:

$$\mathbf{N} = aF\sin\theta\hat{\mathbf{x}}$$

The magnitude of the force on each of these segments is

$$F = IbB$$

and therefore

$$\mathbf{N} = IabB\sin\theta\hat{\mathbf{x}} = mB\sin\theta\hat{\mathbf{x}}$$

or

$$\boxed{\mathbf{N} = \mathbf{m}\times\mathbf{B}} \tag{6.1}$$

where $m = Iab = IA$ is the magnetic dipole moment of the loop. This gives the exact torque on any localized current distribution, in the presence of a *uniform*field; in a *nonuniform* field it is the exact torque (about the center) for a *perfect* dipole of infinitesimal size. Notice that this is analogous to the torque given in electrostatics. In particular, the torque is again in such a direction as to line the dipole *parallel* to the field. It is this torque that accounts for **paramagnetism**. Paramagnetism normally occurs in atoms or molecules with an odd number of electrons, where the "extra" unpaired electron is subject to the magnetic torque.

In a *uniform* field, the *net* force on a current loop is zero:

$$\mathbf{F} = I\oint(d\mathbf{l}\times\mathbf{B}) = I\left(\oint d\mathbf{l}\right)\times\mathbf{B} = 0$$

the constant **B** comes outside the integral, and the net displacement $\oint d\mathbf{l}$ around a closed loop vanishes. For an *infinitesimal* loop, with dipole moment **m**, in a field **B**, the net force is

$$\boxed{\mathbf{F} = \nabla(\mathbf{m}\times\mathbf{B})} \tag{6.2}$$

which is analogous to the its electric counterpart $\mathbf{F} = \nabla(\mathbf{p}\times\mathbf{E})$.

6.3 Effect of a Magnetic Field on Atomic Orbits

Electrons not onl;y *spin*; they also *revolve* around the nucleus- for simpliciuty, let's assume the orbit is a circle of radius R. Although technically this orbital motion does not constitute a steady current, in practice the period $T = 2\pi R/v$ is so short that unless you blink awfully fast, its going to *look* like a steady current

$$I = \frac{e}{T} = \frac{ev}{2\pi R}$$

Accordingly, the orbital dipole moment $(I\pi R^2)$ is

$$\mathbf{m} = -\frac{1}{2}evR\hat{\mathbf{z}}$$

(The minus sign accounts for the negative charge of the electron). Like any other magnetic dipole, this one is subject to a torque $(\mathbf{m}\times\mathbf{B})$ when the atom is placed in a magnetic field. The orbital contribution to paramagnetism is small. There is, however, a more significant effect on the orbital motion: The electron *speeds up* or *slows down*, depending on the orientation of **B**. For wheraa the centripetal acceleration v^2/R is ordinarily sustained by electric forces alone

$$\frac{1}{4\pi\varepsilon_0}\frac{e^2}{R^2} = m_e\frac{v^2}{R}$$

in the presence of a magnetic field there is an additional force, $-e(\mathbf{v}\times\mathbf{B})$. For the sake of argument, let's say that **B** is perpendicular to the plane of orbit, as shown in figure 6.3, then

$$\frac{1}{4\pi\varepsilon_0}\frac{e^2}{R^2} + e\bar{v}B = m_e\frac{v^2}{R}$$

Under these conditions, the new speed $\bar{v}$ is greater than v

$$e\bar{v}B = \frac{m_e}{R}(\bar{v}^2 - v) = \frac{m_e}{R}(\bar{v}+v)(\bar{v}-v)$$

or assuming the change $\Delta v = \bar{v} - v$ is small

$$\Delta v = \frac{eRB}{2m_e}$$

When **B** is turned on, the electron speeds up.

A change in orbital speed means a change in the dipole moment

$$\Delta\mathbf{m} = -\frac{1}{2}e(\Delta v)R\hat{\mathbf{z}} = -\frac{e^2R^2}{4m_e}\mathbf{B} \tag{6.3}$$

Notice that the change in **m** is opposite the direction of **B.** (An electron circling the other way would have a dipole moment pointing *upward*, but such an orbit would be slowed down by the field.

6.4 Magnetization and the Field of a Magnetized Object

In the presence of a magnetic field, matter becomes *magnetized*. Upon a microscopic examination it will be found to contain many tiny dipoles, with a net alignment along some direction. We describe the state of magnetic polarization by the vector quantity

$$\mathbf{M} \equiv \textit{magnetic dipole moment per unit volume}$$

where **M** is called the **magnetization**; it plays a role analogous to the polarization **P** in electrostatics. In the following section, we will not worry about how the magnetization got there- it could be paramagnetism, diamagnetism, or even ferromagnetism- we shall take **M** as *given*, and calculate the field this magnetization itself produces.

6.4.1 Bound Currents

Suppose we have a piece of magnetized material; the magnetic dipole moment per unit volume, **M**, is given. What field does this produce? Well, the vector potential of a single dipole **m** is given by Equation 5.12

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r'^2}$$

where r' is the seperation vector. In the magnetized object, each volume element $d\tau'$ carries a dipole moment $\mathbf{M}d\tau'$, so that the total vector potential

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{M}(\mathbf{r}')\times\hat{\mathbf{r}}}{r'^2}d\tau'$$

That *does* it, in principle. But as in the electrical case, the integral can be cast in a more illuminating form by exploiting the identity

$$\nabla'\frac{1}{r'} = \frac{\hat{\mathbf{r}'}}{r'^2}$$

with this,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int\left[\mathbf{M}(\mathbf{r}') \times \left(\nabla'\frac{1}{r'}\right)\right] d\tau'$$

Integrating by parts, uisng product rule 7, gives

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\left[\int \frac{1}{r'}[\nabla' \times \mathbf{M}(\mathbf{r}')]d\tau' - \int \nabla' \times \left[\frac{\mathbf{M}(\mathbf{r}')}{r'}\right] d\tau'\right]$$

We can write the latter as a surface integral

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{1}{r'}[\nabla' \times \mathbf{M}(\mathbf{r}')]d\tau' - \frac{\mu_0}{4\pi}\int \frac{1}{r'}[\mathbf{M}(\mathbf{r}') \times d\mathbf{a}']$$

The first term looks just like the potential of a *volume* current

$$\boxed{\mathbf{J}_b = \nabla \times \mathbf{M}} \tag{6.4}$$

while the second term looks like the potential of a surface current,

$$\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}} \tag{6.5}$$

where $\hat{\mathbf{n}}$ is the normal vector. With these definitions,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}_b(\mathbf{r}')}{r'}d\tau' + \frac{\mu_0}{4\pi}\int_S \frac{\mathbf{K}_b(\mathbf{r}')}{r'}da'$$

What this means is that the potential (and hence also the field) of a magnetized object is the same as would be produced by a volume current $\mathbf{J}_b = \nabla \times \mathbf{M}$ throughout the material, plus a surface current $\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}}$,on the boundary.

6.5 The Auxiliary Field H

6.5.1 Ampere's law in Magnetized Materials

In the previous section we found that the effect of magnetization is to establish bound currents $\mathbf{J}_b = \nabla \times \mathbf{M}$ within the material and $\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}}$ on the surface. We are now ready to put everything together: the field attributable to bound currents, plus the field due to everything else- which I shall conductocall **free current**. The free current might flow through wires imbedded in the magnetized substance or, if the latter is a conductor, through the material itself. In any event, the total current can be written as

$$\mathbf{J} = \mathbf{J}_b + \mathbf{J}_f \tag{6.6}$$

In view of Equation 6.4 and 6.6, Ampere's law can be written

$$\frac{1}{\mu_0}(\nabla \times \mathbf{B}) = \mathbf{J} = \mathbf{J}_b + \mathbf{J}_f = \mathbf{J}_f + (\nabla \times \mathbf{M})$$

or, collecting together the two curls

$$\nabla \times \left(\frac{1}{\mu_0}\mathbf{B} - \mathbf{M}\right) = \mathbf{J}_f$$

The quantity in paranthesis is designated by the letter **H**:

$$\mathbf{H} \equiv \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \tag{6.7}$$

In terms of **H**, then, Ampere's law reads

$$\nabla \times \mathbf{H} = \mathbf{J}_f \tag{6.8}$$

or, in integral form,

$$\oint \mathbf{H} \cdot d\mathbf{l} = I_{f\text{enc}} \tag{6.9}$$

where $I_{f_{enc}}$ is the total $free$ current passing through the Amperian loop.

H plays a role analogous to **D** in electrostatics: Just as **D** allowed us to write $Gauss's$ law in terms of the free *charge* alone, **H** permits us to express $Ampere's$ law in terms of the free *current* alone- and free current is what we control. Bound current, like bound charge, comes along for the ride- the material gets magnetized, and this results in bound currents

6.5.2 Boundary Conditions

The divergence of **H** is given by

$$\nabla \cdot \mathbf{H} = -\nabla \cdot \mathbf{M} \tag{6.10}$$

The magnetostatics boundary conditions can ne written in terms of **H** and the $free$ current. From Equation 6.10 it follows that

$$H^{\perp}_{\text{above}} - H^{\perp}_{\text{below}} = -(M^{\perp}_{\text{above}} - M^{\perp}_{\text{below}})$$

while Equation 6.8 says

$$\mathbf{H}^{\|}_{\text{above}} - \mathbf{H}^{\|}_{\text{below}} = \mathbf{K}_f \times \hat{\mathbf{n}}$$

In the presence of materials these conditions are sometimes more useful then the corresponding boundary conditions on **B**

$$B^{\perp}_{\textbf{above}} - B^{\perp}_{\textbf{below}} = 0$$

and

$$\mathbf{B}^{\|}_{\text{above}} - \mathbf{B}^{\|}_{\text{below}} = \mu_0(\mathbf{K} \times \hat{\mathbf{n}})$$

6.6 Linear and Nonlinear Media

In paramagnetic and diamagnetic materials, the magnetization is sustained by the field; when **B** is removed, **M** disappears. In fact, for most substances the magnetization is $proprtional$ to the field, provided the field is not too strong.

$$\mathbf{M} = \chi_m \mathbf{H} \tag{6.11}$$

The constant of proportionality χ_m is called the **magnetic susceptibility**; it is a dimensionless quantity that variues from one substance to another- positive for paramagnets and negative for diamagnets. Materials that obey the above relationship are called **linear media**. We also know

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) = \mu_0(1 + \chi_m)\mathbf{H}$$

for linear media. Thus **B** is also proportional to **H**

$$\mathbf{B} = \mu \mathbf{H}$$

where

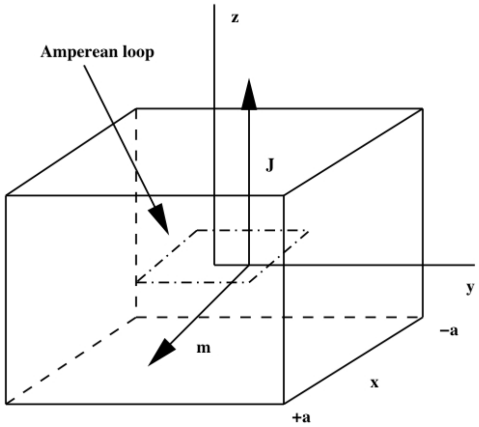
$$\mu \equiv \mu_0(1+\chi_m)$$

μ is called the **permiability** of the material. In vacuum, where there is no matter to magnetize, the susceptibility χ_m vanishes, and the permiability is μ_0. That's why μ_0 is called the **permiability of free space**.

6.7 Problems and Solutions

Problem # 1 Forces on magnetic dipoles

A uniform current density $\mathbf{J} = J_0\hat{\mathbf{z}}$ fills a slab straddling the yz plane, from $x = -a$ to $x = +a$. A magnetic dipole $\mathbf{m} = m_0\hat{\mathbf{x}}$ is situated at the origin.



a). Find the force on the dipole , using Equation 1.

The force on a magnetic diple is given as

$$\mathbf{F} = \nabla(\mathbf{m}\cdot\mathbf{B}) \tag{6.12}$$

We need to use Gauss's law and an Amperean loop to find what the magnetic field is, we can see that the magnetic field will point in the $\hat{\mathbf{y}}$ dircetion, We know it cannot point in the $\hat{\mathbf{z}}$ direction or the $\hat{\mathbf{x}}$ direction.

$$\int \mathbf{B}\cdot d\mathbf{l} = \mu_0 \int \mathbf{J}\cdot d\mathbf{a}$$

we know that $d\mathbf{l} = y\hat{\mathbf{y}}$ and $d\mathbf{a} = xy\hat{\mathbf{z}}$, thus

$$\mathbf{B} = \mu_0 J_0 x\hat{\mathbf{y}}$$

and so we find that the force is

$$\mathbf{F} = \nabla(\mathbf{m}\cdot\mathbf{B}) = \boxed{0}$$

because **m** and **B** are orthogonal.

b). Do the same for the dipole pointing in the $y-$direction: $\mathbf{m} = m_0\hat{\mathbf{y}}$.

We find that the force with this new dipole orientation is given by

$$\mathbf{F} = \nabla(\mathbf{m}\cdot\mathbf{B}) = \nabla(m_0\hat{\mathbf{y}}\cdot\mu_0 J_0 x\hat{\mathbf{y}}) = \frac{\partial}{\partial x}(m_0\mu_0 J_0 x)\hat{\mathbf{x}} = \boxed{m_0\mu_0 J_0\hat{\mathbf{x}}}$$

c). In the *elesctrostatic* case the expression $\mathbf{F} = \nabla(\mathbf{p}\cdot\mathbf{E})$ and $\mathbf{F} = (\mathbf{p}\cdot\nabla)\mathbf{E}$ are equivalent (prove it), but this is not the case for the magnetic analogs (explain why). As an example calculate $(\mathbf{m}\cdot\nabla)\mathbf{B}$ for the configurations in (a) and (b).

We know that in electrostatics this statement is true

$$\nabla(\mathbf{p}\cdot\mathbf{E}) = (\mathbf{p}\cdot\nabla)\mathbf{E}$$

we know that we can expand the left hand term using Equation 4 in the from of Griffith's, suing product rules, i.e

$$\nabla(\mathbf{p}\cdot\mathbf{E}) = \mathbf{p}\times(\nabla\times\mathbf{E}) + \mathbf{E}\times(\nabla\times\mathbf{p}) + (\mathbf{p}\cdot\nabla)\mathbf{E} + (\mathbf{E}\cdot\nabla)\mathbf{p}$$

but we know

$$\mathbf{p}\times(\nabla\times\mathbf{E}) = \mathbf{E}\times(\nabla\times\mathbf{p}) = (\mathbf{E}\cdot\nabla)\mathbf{p} = 0$$

the first term goes to 0 by definition, if **E** is a uniform constant electric field, the second term goes to 0 because **p** is also a contstant vector, in a linear dialectric medium **p** is proportional to **E**. The last term also goes to 0 because **p** is a costant vector, and when you take the derivative of a constant vector you get 0. Thus

$$\nabla(\mathbf{p}\cdot\mathbf{E}) = (\mathbf{p}\cdot\nabla)\mathbf{E} \qquad \text{QED}$$

The magnetic analog is given as

$$\nabla(\mathbf{m}\cdot\mathbf{B}) = \mathbf{m}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{m}) + (\mathbf{m}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{m}$$

but we also know that

$$\mathbf{B}\times(\nabla\times\mathbf{m}) = (\mathbf{B}\cdot\nabla)\mathbf{m} = 0$$

and

$$\mathbf{m}\times(\nabla\times\mathbf{B}) \neq 0$$

thus we find

$$\nabla(\mathbf{m}\cdot\mathbf{B}) = \mathbf{m}\times(\nabla\times\mathbf{B}) + (\mathbf{m}\cdot\nabla)\mathbf{B}$$

for the configurations in a) we find

$$(\mathbf{m}\cdot\nabla)\mathbf{B} = m_0\frac{\partial}{\partial x}(\mu_0 J_0 x)\hat{\mathbf{y}} = \boxed{m_0\mu_0 J_0\hat{\mathbf{y}}}$$

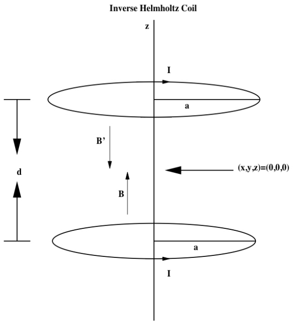
and for the configuration in b) we find

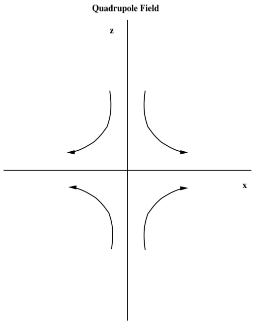
$$(\mathbf{m}\cdot\nabla)\mathbf{B} = m_0\frac{\partial}{\partial y}(\mu_0 J_0 x)\hat{\mathbf{y}} = \boxed{0}$$

The next set of problems concern the magnetic trapping of ultracold atoms, specifically atoms of ^{87}Rb (one of the rubidium isotopes). In one of their internal states (the F = 1 hyperfine state), such atoms have a magnetic dipole moment of $m = m_B/2$ where $m_B = e\hbar/2m_e$ is the Bohr magneton. Magnetic trapping is achieved under the condition that the atomic magnetic moment remains anti-aligned with the magnetic field as the atom moves around in space, thus producing a trapping potential of the form $U(\mathbf{r}) = m|B|$.

Problem # 2 Spherical quadrupole magnetic trap

First, let's consider a simple kind of magnetic trap known as the spherical quadrupole magnetic trap.





a) Consider two electromagnet coils sharing the same axis and separted by d, each of radius a and N turns, each coil carrying current I running in the opposite direction as that of the other coil. Placing the origin at the midpoint between the two coils, calculate the magnetic field $B(x,y,z)$ in the region near the origin, keeping just the lowest order terms. Note that to determine the field for radial displacements (i.e. in the x or y directions), you can consider the symmetry of the problem and apply the condition of zero-divergence for the magnetic field.

An expansion about the origin of the magnetic field $B(x,y,z)$ yields

$$\begin{aligned}
B_z(x,y,z) &= B_0 + f'(B_z) = ax + by + cz \\
B_y(x,y,z) &= B_0 + f'(B_y) = a'x + b'y + c'z \\
B_x(x,y,z) &= B_0 + f'(B_x) = a''x + b''y + c''z
\end{aligned}$$

we know that B_0 is 0 at the origin. We also know that

$$a = b = b = a' = c' = b'' = c'' = 0$$

thus we can write the expanded field as

$$B(x,y,z) = a''x + b'y + cz$$

and since we know that $\nabla \cdot \mathbf{B} = 0$ we find

$$\frac{\partial}{\partial x}a''(x) + \frac{\partial}{\partial y}b'(y) + \frac{\partial}{\partial z}c(z) = 0$$

but since we know that this configuration has rotational symmetry we can say

$$\frac{\partial}{\partial x}(x) = \frac{\partial}{\partial y}(y) \qquad a'' = b'$$

we find

$$c = -2a''$$

now we just need to find out what c is. From our expansion we can see that

$$f'(B_z) = cz$$

and we know that the magnetic field from a loop of wire at some point z away from the loop is given by

$$B(z) = \frac{\mu_0 I}{2}\frac{a^2}{(a^2+z^2)^{3/2}}$$

but for our configuration, and the fact that we have two loops with currents in opposite directions we find

$$B(z) = \frac{\mu_0 I}{2}\left[\frac{a^2}{(a^2+(z-d/2)^2)^{3/2}} - \frac{a^2}{(a^2+(z+d/2)^2)^{3/2}}\right]$$

and we know that

$$f'(B(z)) = \frac{\mu_0 I}{2}\left[-\frac{3a^2(z-d/2)}{(a^2+(z-d/2)^2)^{5/2}} + \frac{3a^2(z+d/2)}{(a^2+(z+d/2)^2)^{5/2}}\right]$$

but since we want to know what the magnetic field is at the origin we find

$$f'(B(z=0)) = \frac{3\mu_0 I a^2}{2}\left[\frac{d}{(a^2+(d/2)^2)^{5/2}}\right]$$

thus we find that

$$c = \frac{3\mu_0 I a^2}{2}\left[\frac{d}{(a^2+(d/2)^2)^{5/2}}\right]$$

and

$$a'' = b' = -\frac{c}{2} = -\frac{3\mu_0 I a^2}{4}\left[\frac{d}{(a^2+(d/2)^2)^{5/2}}\right]$$

thus the magnetic field at the origin is given by

$$B(x,y,z) = \boxed{-\frac{3\mu_0 I N a^2}{4}\left[\frac{d}{(a^2+(d/2)^2)^{5/2}}\right](x\hat{\mathbf{x}}+y\hat{\mathbf{y}}-2z\hat{\mathbf{z}})}$$

b) You should find a spherical quadrupole field with field gradients in the radial and axial directions determined by properties of the coils which create the field. A spherical quadrupole field can be written as

$$B(x,y,z) = B'(x\hat{\mathbf{x}}+y\hat{\mathbf{y}}-2z\hat{\mathbf{z}})$$

where B' is the radial field gradient. Now put in some real numbers. Consider coils of radius a = 2.5 cm separated by $d = 2a/\sqrt{3}$ (the optimal separation), each of 20 turns running I = 92 amperes. What are the axial and radial gradients (G/cm)?

We know that the radial field gradient is given by

$$B'_{radial} = \frac{3\mu_0 I N a^2}{4}\left[\frac{d}{(a^2+(d/2)^2)^{5/2}}\right] = \boxed{1561\text{G/cm}}$$

and the axial field gradient is

$$B'_{axial} = -\frac{3\mu_0 INa^2}{2}\left[\frac{d}{(a^2+(d/2)^2)^{5/2}}\right] = \boxed{-3120\text{G/cm}}$$

Problem # 3 Majorana losses

In the spherical quadrupole magnetic field described above, the potential $U(\mathbf{r}) = m|B|$ is clearly minimized at a point, and, thus, particles with their magnetic moment anti-aligned with the field will be trapped by such a field. However, a problem with the spherical quadrupole trap is the fact that it is always lossy. Magnetic trapping requires that the atomic magnetic moment always follow the local orientation of the magnetic field. If the field is changing its direction faster than an atom's orientation can keep up, the atomic orientation may change, causing the atom to be ejected from a magnetic trap. This type of loss is given the name Majorana loss.

a) Consider a ^{87}Rb atom moving with velocity $v$and impact parameter b in the $x-y$ plane of a spherical quadrupole trap. Give an estimate for the critical impact parameter b_0 below which the atom has a high probability of being lost, and above which the atom has a low probability of being lost. To do this, consider the maximal rate at which the $orientation$ of the field is changing - this quantity has units of s^{-1}. Compare this with the Larmor precession frequency of the atom in the local magnetic field (this frequency is zero at the origin and larger elsewhere).

We know that the orbital frequency and Larmar frequency for this problem are given by

$$\omega_{orb} = \frac{v}{b_0} \qquad \omega_{Lar} = \gamma B$$

setting these two expressions equal to each other yields

$$\frac{v}{b_0} = \gamma B = \gamma B' b_0$$

but we know that

$$\gamma = \frac{Q}{2m}$$

thus we find

$$b_0 = \boxed{\sqrt{\frac{2mv}{QB'}}}$$

b) Now we will estimate the Majorana loss rate for atoms at a certain temperature trapped in a spherical quadrupole trap. To do this, consider that the treatment above determines an ellipsoidal volume within which atoms at a given velocity may be lost from the trap. A loss rate can be calculated as the product $n\sigma v$ where n is the density of the atomic gas, σ is the area of the ellipsoid (which differs slightly if approached from different directions; just take some characteristic area), and v is the typical velocity of atoms. To estimate the volume occupied by the gas (you'll need this to determine the density n), consider how large is the volume in which the trap potential $U(\mathbf{r})$ is smaller than $k_B T$where k_B is the Boltzmann constant. Similarly, to obtain a typical velocity, consider the velocity at which the kinetic energy is $k_B T$. Having made all these estimates, find the loss rate per atom given the temperature T and the radial field gradient B' .

Since we know that the lost rate is given by

$$L_{rate} = n\sigma v$$

and we know that the energy is given by

$$U(\mathbf{r}) = \mathbf{m}|B| = \mathbf{m}|B'(x\hat{\mathbf{x}} + y\hat{\mathbf{y}})| = \mathbf{m}B'r$$

and if the thermal energy is equal to the potential energy we get

$$k_B T = \mathbf{m}B'r$$

rearranging this for r gives

$$r = \frac{k_B T}{\mathbf{m}B'}$$

and the volume is given by

$$V = \frac{4}{3}\pi r^3 - \frac{4}{3}\pi b_0^3$$

but since

$$b_0 \ll r$$

we find the volume to be given as

$$V = \frac{4}{3}\pi r^3$$

and using our expression for r we find that the volume is given by

$$V = \frac{4}{3}\pi\left(\frac{k_B T}{\mathbf{m}B'}\right)^3$$

and we know that the loss rate is given by

$$L_{rate} = n\sigma v = \frac{N}{V}\sigma v$$

we know that the velocity can be found using

$$\frac{1}{2}mv^2 = k_B T \quad v = \sqrt{\frac{2k_b T}{m}}$$

and also

$$\sigma = \pi b_0^2$$

thus

$$L_{rate} = \boxed{\frac{3b_0^2\sqrt{\frac{2k_b T}{m}}}{4\left(\frac{k_B T}{\mathbf{m}B'}\right)^3} = \frac{3N\mu_{\mathbf{B}}{}^3(B')^2}{Q(k_B T)^2}}$$

c) Again, let's put in some numbers. Consider a gas of ^{87}Rb trapped in a spherical quadrupole trap with a radial gradient of 120 G/cm. What is the Majorana loss rate per atom at temperatures of 1, 10, and 100 μK.

Using

$$\frac{L_{rate}}{N} = \left[\frac{3\mu_B^3(B')^2}{Q(k_B T)^2}\right]$$

we find

$$\begin{aligned} L_{rate} &= 3743\ \frac{\text{atoms}}{\text{s}} \quad T = 1\mu\text{K} \\ L_{rate} &= 37.4\ \frac{\text{atoms}}{\text{s}} \quad T = 10\mu\text{K} \\ L_{rate} &= 0.374\ \frac{\text{atoms}}{\text{s}} \quad T = 100\mu\text{K} \end{aligned}$$

Problem # 4 The TOP trap

Losses from spherical quadrupole traps made evaporative cooling of atoms in magnetic traps ineffective until some manner was employed to avoid these losses. As you found above, the problem becomes ever worse as the temperature of the gas gets lower, thus precluding the ultra-low temperatures needed to reach Bose-Einstein condensation. One solution, employed by the team of Cornell and Wieman in Colorado for the first achievement of Bose-Einstein condensation in a gas, was the TOP (time-averaged orbiting potential) trap. Consider that a spherical quadrupole trap is formed, and then an additional bias field is applied of the form

$$B_{TOP}(x,y,z) = B_0 cos\omega t\hat{\mathbf{x}} + B_0 sin\omega t\hat{\mathbf{y}}$$

Atoms held in a time-varying magnetic trap of this type will experience a time-averaged potential at every point so long as (1) the field rotates at a rate faster than the motional time-scale of atoms so that an atom experiences a local average of the magnitude of the field, and (2) the field rotates at a rate slower than the Larmor precession frequency so that atomic magnetic moment remains anti-aligned with the magnetic field.

a) Assume that these conditions are met. What is the time-averaged magnitude of the magnetic field near the origin? You should find that this magnitude has a minimum at the origin, and varies quadratically away from the origin with a fixed ratio of trap curvatures in the different directions.

We know that the total magnetic field is given by

$$\begin{aligned} \mathbf{B}_{tot} &= B'(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} - 2z\hat{\mathbf{z}}) + B_0 cos\omega t\hat{\mathbf{x}} + B_0 sin\omega t\hat{\mathbf{y}} \\ &= (B'x + B_0\cos\omega t)\hat{\mathbf{x}} + (B'y + B_0\cos\omega t)\hat{\mathbf{y}} - 2B'z\hat{\mathbf{z}} \end{aligned}$$

and the magnitude is given by

$$B_{tot} = \sqrt{(B'x)^2 + (B'y)^2 + (2B'z)^2 + 2B'B_0(x\cos\omega t + y\sin\omega t)}$$

thus magnitude of the time-averaged magnetic field is given by

$$\langle B_{tot}\rangle = \frac{\omega}{2\pi}\int_0^{2\pi/\omega}\sqrt{(B'x)^2 + (B'y)^2 + (2B'z)^2 + 2B'B_0(x\cos\omega t + y\sin\omega t)}dt$$

This can also be written as

$$\langle B_{tot}\rangle = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} B_0\sqrt{1+\varepsilon}dt$$

where ε is given as

$$\varepsilon = \left(\frac{B'}{B_0}\right)^2 (x^2 + y^2 + 4z^2) + \frac{2B'}{B_0}(x\cos\omega t + y\sin\omega t)$$

we can now Taylor expand about the origin to find

$$\sqrt{1+\varepsilon} = 1 + \frac{\varepsilon}{2} - \frac{\varepsilon^2}{8} + \ldots$$

and we know that

$$\varepsilon^2 = \left[\left(\frac{B'}{B_0} \right)^2 (x^2 + y^2 + 4z^2) + \frac{2B'}{B_0} (x \cos \omega t + y \sin \omega t) \right]^2$$

and we know that the only thing that survives from this in the time averaged field is

$$\varepsilon^2 \approx \left[\frac{2B'}{B_0} (x \cos \omega t + y \sin \omega t) \right]^2$$

where the first higher order terms in x, y, and z vanish in ε^2. We can write the time averaged expanded field as (after integration)

$$\langle B_{tot} \rangle = \boxed{B_0 + \frac{B'^2}{2B_0}(x^2 + y^2 + 4z^2) - \frac{B'^2}{4B_0} x^2 - \frac{B'^2}{4B_0} y^2}$$

b) Now put in some numbers. Suppose on top of the 120 G/cm radial gradient spherical quadrupole field, the time-varying TOP field has a magnitude of 10 G and rotates at a rate of $\omega = 2\pi \times 7.5$ kHz. What are the oscillation frequencies of a ^{87}Rb atom in this trap? (Note: if a one dimensional potential has the form $U(x) = (1/2)m\Omega^2 x^2$, then a particle of mass m will oscillate harmonically in this potential with oscillation angular frequency Ω). Do the field settings chosen satisfy the two conditions for adiabaticity described above?

We know that the potential energy in three dimensions can be written as

$$U(r) = \frac{1}{2} m \Omega^2 r^2 = \mu_B B$$

where B is the time averaged magnetic field from part a). Thus we find that the oscillation frequency can be expressed as

$$\begin{aligned} \Omega_x &= \left(\frac{B'}{B_0} \right) \sqrt{\frac{\mu_B B_0}{2 m_{Rb}}} = 151 \frac{1}{\text{s}} \\ \Omega_y &= \left(\frac{B'}{B_0} \right) \sqrt{\frac{\mu_B B_0}{2 m_{Rb}}} = 151 \frac{1}{\text{s}} \\ \Omega_z &= \left(\frac{B'}{B_0} \right) \sqrt{\frac{4 \mu_B B_0}{m_{Rb}}} = 428 \frac{1}{\text{s}} \end{aligned}$$

c) Finally, the critical temperature T_c for Bose-Einstein condensation for atoms in a harmonic trap is determined by the geometric mean of trapping frequencies $\bar{\omega}$ and the number of atoms N by the relation

$$N = \xi(3) \left(\frac{k_B T_c}{\hbar \bar{\omega}} \right)^3$$

where k_B is the Boltzmann constant and ξ is the Riemann Zeta function ($\xi(3)$ =1.202). This equation can be interpreted as stating that once the number of quantum states available to a gas at a temperature

T_c roughly equals the number of particles, multiple occupancy of quantum states occurs and quantum statistical effects such as Bose-Einstein condensation ensue. Consider the above TOP trap with 2×10^4 atoms. What is the Bose-Einstein condensation transition temperature?

The critical temperature can be expressed as

$$T_c = \frac{N^{1/3}\hbar\bar{\omega}}{k_B\xi(3)^{1/3}}$$

where $\bar{\omega}$ is defined as the geometric mean

$$\bar{\omega} = (\Omega_x\Omega_y\Omega_z)^{1/3} = 213\frac{1}{\text{s}}$$

thus

$$T_c \approx \boxed{41.6\text{ nK}}$$

Problem # 5 Refrigerator magnets and atom mirrors

Most of you have probably seen large, flexible refrigerator magnets. These are flat magnetic materials, up to a couple of inches on a side, that stick to an iron refrigerator door by inducing magnetization in the door with the magnetic fields emanating from the surface of the magnet.

a) Suppose you have such a magnet with a circular shape, and that magnet is uniformly magnetized, with a magnetization out of the plane. What is the magnetic field on the surface of the magnet at its center, as a function of the radius of the magnet? On the basis of this observation, explain why this is not how refrigerator magnets are magnetized.

Since we are assuming that this is a ring of charge then we already know what the magnetic field is, we have seen this problem before, the magnetic field of a ring of charge is given by

$$\mathbf{B}(z) = \frac{\mu_0 I}{2}\frac{R^2}{(R^2+z^2)^{3/2}}\hat{\mathbf{z}}$$

but since we are only concerned about the magnetic field at $z = 0$ we find that the magnetic field is given by

$$\mathbf{B}(R) = \frac{\mu_0 I}{2R}\hat{\mathbf{z}}$$

This would not make a very good magnetic due to the fact that as the radius increases the magnetic field decreases. If we say that the thickness of the magnetic is h and also a magnetization of M we can write the current as

$$I = Mh = K_b h$$

and find that the magnetic field due to this configuration is given by

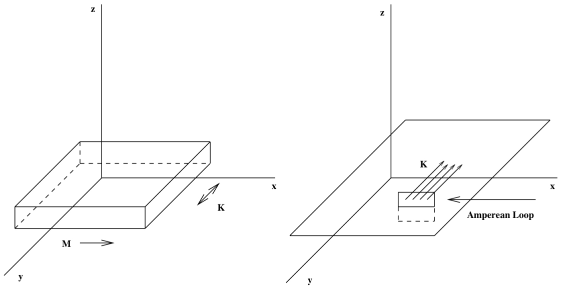
$$\mathbf{B}(r) = \frac{\mu_0 Mh}{2R}\hat{\mathbf{z}} = \frac{\mu_0 K_b h}{2R}\hat{\mathbf{z}}$$

and this field has the same charactersistics. Thus if we have a magnet that is twice as big, the magnetic field will be twice as small.

The magnetization in such a refrigerator magnet is actually a periodic linear array of oppositely oriented magnetization. Let us consider the field produced by such a magnetization. Specifically, consider a magnet that fills the space $-d < z < 0$, being infinite in extent in the $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ directions. Let the magnetization $\mathbf{M}$ be oriented in the plane of the magnet, varying as

$$\mathbf{M} = Msinkx\hat{\mathbf{x}}$$

where $\mathbf{M}$ gives the maximum value of the magnetization and k the spatial wavevector of its modulation.



b) Show that this magnetization can be represented by bound surface currents at $z = 0$ (top layer) and $z = -d$ (bottom layer).

We know that

$$K_b = \mathbf{M} \times \hat{\mathbf{n}}$$

so for the top surface

$$K_b(z=0) = \mathbf{M} \times \hat{\mathbf{z}} = M \sin kx(-\hat{\mathbf{y}}) \qquad \hat{\mathbf{n}} = \hat{\mathbf{z}}$$

and at the bottom

$$K_b(z=-d) = \mathbf{M} \times \hat{\mathbf{z}} = M \sin kx(\hat{\mathbf{y}}) \qquad \hat{\mathbf{n}} = -\hat{\mathbf{z}}$$

c) Consider just the field due to the top layer of bound currents. In the region $z = 0$, there are no currents, and therefore $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{B} = 0$. Thus, we can define a scalar magnetic potential V_m for that obeys $\nabla^2 V_m = 0$ and $\mathbf{B} = -\nabla V_m$. Considering these relations, the boundary condition at $z \to \infty$, and the y-independence of this problem, show that V_m can be written as

$$V_m(\mathbf{r}) = \pm V_0 \cos(kx + \phi) e^{-k|z|}$$

where the ± refer to solutions for V_m either above or below the layer of currents. The values of V_0 and ϕ are still to be determined. Hint: recall solutions to the Laplace equation that are separated by Cartesian coordinates.

We can solve this problem by using seperation of variables in cartesian coordinates

$$V_m = X(x)Z(z)$$

we need to solve Laplaces equation

$$\nabla^2 V_m = \frac{\partial^2 V_m}{\partial x^2} + \frac{\partial^2 V_m}{\partial y^2} = 0$$

which becomes

$$\frac{\partial^2 X}{\partial x^2} = -k^2 X(x) \quad \frac{\partial^2 Z}{\partial z^2} = k^2 Z(z) \qquad k \geq 0$$

which is just a second order differential equation with a decaying solution and an oscillating solution, i.e

$$X(x) = A\cos kx + B\sin kx \qquad Z(z) = Ce^{kz} + De^{-kz}$$

and our boundary condition $z \to \infty$ $V_m \to 0$ gives

$$C = 0 \quad A\cos(kx) + B\sin(kx) = A'\cos(kx+\phi)$$

where we find

$$X(x) = A'\cos(kx+\phi) \quad Z(z) = De^{-kz}$$

and we can finally write the scalar potential as

$$\begin{aligned} V_m &= V_0\cos(kx+\phi)e^{-k|z|} \quad \text{above the surface} \\ V_m &= -V_0\cos(kx+\phi)e^{-k|z|} \quad \text{below the surface} \end{aligned}$$

where V_0 absorbs A' and D.

d) Now obtain an expression for the magnetic fields from the top layer of currents, still containing unknown quantities V_0 and ϕ. Use Ampere's law to determine those quantities and thus to fully determine the fields.

We know that the magnetic field can be found with

$$B(z>0) = -\nabla V_m$$

which can also be written as

$$\mathbf{B}(x,z>0) = -\left(\frac{\partial V_m}{\partial x}\hat{\mathbf{x}} + \frac{\partial V_m}{\partial z}\hat{\mathbf{z}}\right) = kV_0(\sin(kx+\phi)e^{-kz}\hat{\mathbf{x}} + \cos(kx+\phi)e^{-kz}\hat{\mathbf{z}})$$

and for below the surface we know that the magnetic field can be found with

$$B(z<0) = -\nabla V_m$$

which can also be written as

$$\mathbf{B}(x,z<0) = -\left(\frac{\partial V_m}{\partial x}\hat{\mathbf{x}} + \frac{\partial V_m}{\partial z}\hat{\mathbf{z}}\right) = kV_0(-\sin(kx+\phi)e^{kz}\hat{\mathbf{x}} + \cos(kx+\phi)e^{kz}\hat{\mathbf{z}})$$

to find what V_0 and ϕ is we can use the integral form of Ampere's law, which is given by

$$\int \mathbf{B}\cdot d\mathbf{l} = \mu_0 I_{enc}$$

but we know that $d\mathbf{l} = dx\hat{\mathbf{x}}$ on the top of the loop and $I_{enc} = K_b dx$ thus we find, we know that the verticle line segments cancel, thus we find

$$2\int_0^l V_0(k\sin(kx+\phi)\hat{\mathbf{x}} - k\cos(kx+\phi)\hat{\mathbf{z}})\cdot dx\hat{\mathbf{x}} = \mu_0\int_0^l M\sin kx dx$$

which becomes

$$2kV_0\int(\sin kx+\phi)dx = \mu_0 M\int \sin kx dx$$

thus we can see that

$$V_0 = \frac{\mu_0 M}{2k} \quad \phi = 0$$

thus the magnetic field on the top layer is

$$\begin{aligned}\mathbf{B}(x,z &> 0) = \boxed{\frac{\mu_0 M}{2} e^{-kz}(\sin(kx)\hat{\mathbf{x}} + \cos(kx)\hat{\mathbf{z}})} \\ \mathbf{B}(x,z &< 0) = \boxed{\frac{\mu_0 M}{2} e^{kz}(-\sin(kx)\hat{\mathbf{x}} + \cos(kx)\hat{\mathbf{z}})}\end{aligned}$$

e) So far you have determined the field from the top layer. Now add the fields from the bound currents on both layers of the magnet. What is the magnitude of the field on the surface of the magnet ($z = 0$)?

We know that the contribution to the magnetic field from the bottom layer is going to be equal and opposite as for the top layer (current is running in oppoiste direction) than the magnetic fields for the bottom layer are given by

$$\begin{aligned}\mathbf{B}(x,z &> 0) = -\frac{\mu_0 M}{2} e^{-kz}(\sin(kx)\hat{\mathbf{x}} + \cos(kx)\hat{\mathbf{z}}) \\ \mathbf{B}(x,z &< 0) = -\frac{\mu_0 M}{2} e^{kz}(-\sin(kx)\hat{\mathbf{x}} + \cos(kx)\hat{\mathbf{z}})\end{aligned}$$

To find the total magnetic we must add these two

$$\mathbf{B}_T = \mathbf{B}_b + \mathbf{B}_T = \frac{\mu_0 M}{2}(\sin(kx)\hat{\mathbf{x}} - \cos(kx)\hat{\mathbf{z}}) - \frac{\mu_0 M}{2}(\sin(kx)\hat{\mathbf{x}} - \cos(kx)\hat{\mathbf{z}}) = 0$$

Aside from binding to refrigerators, such periodic magnetizations can also be used as mirrors for beams of atoms with magnetic moments. See, for instance, Roach et al., Physical Review Letters 75, 629 (1995). 2)

Problem # 6 The magnetic scalar potential

Maxwell's equations for the magnetic field (Ampere's law, in particular) clearly demonstrate that the magnetic field *cannot* be defined via a scalar potential. Further, there are no magnetic charges, and, hence, no magnetic charge density. Nevertheless, such quantities are useful even though they are solely fictional. Here we consider the use of the magnetic charge density ρ_m and the magnetic scalar potential V_m as devices to determine the fields from magnetized materials.

a) We start with the field of the ideal magnetic dipole. In class, we found out that the electric field of an ideal electric dipole and the magnetic field of an ideal magnetic dipole are quite similar in form, the only difference being the singular field at the position of the dipole. Considering the relation between these two fields, argue why one can write the field from a magnetic dipole as

$$\mathbf{B}(\mathbf{r}) = -\nabla V_m(\mathbf{r}) + \mu_0 M(\mathbf{r}) \tag{6.13}$$

and determine the magnetic scalar potential V_m in this case.

We know that since electric properties are analogous to magnetic properties, then the defenition of the magnetic dipole seems reasonable. We know that we can compare the first term of the magnetic dipole to the first term of the electric dipole. The second term is neccassery so that (like in electrostatics) we can account for the discontinuity at $r = 0$ just like in the case of electrostatics. This is only in the absence of free currents.

Since we know that the magnetic field and electric field have similar characteristics, we can see that the electric field dipole can be expressed as

$$\mathbf{E}_{dip} = \frac{p}{4\pi\varepsilon_0 r^3}(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\theta}) + \frac{\mathbf{p}}{3\varepsilon_0}\delta(r)$$

we can compare the magnetic dipole to this dipole to find

$$\mu_0 M(\mathbf{r}) \quad \text{analogous to} \quad \frac{\mathbf{p}}{3\varepsilon_0}\delta(r)$$

and so

$$\frac{\mu_0 m}{4\pi r^3}(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\theta}) \quad \text{analogous to} \quad \frac{p}{4\pi\varepsilon_0 r^3}(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\theta})$$

thus we can see from this relationship that

$$-\nabla V_m = \frac{\mu_0 m}{4\pi r^3}(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\theta})$$

and since we know that the scalar potential for the electric field dipole is given by

$$V_{dip}(r,\theta) = \frac{1}{4\pi\varepsilon_0}\frac{\hat{\mathbf{r}}\cdot\mathbf{p}}{r^2} = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}$$

and the analogous formula for the magnetic dipole scalar potential is given by

$$V_m(r,\theta) = \frac{\mu_0}{4\pi}\frac{\hat{\mathbf{r}}\cdot\mathbf{m}}{r^2} = \boxed{\frac{\mu_0 m\cos\theta}{4\pi r^2}}$$

b) Now consider the field produced by a magnetization **M**. Using Eq. 1, and following our treatment of the electric field from polarized objects, show that the magnetic scalar potential in this case would be defined as

$$V_m(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_S \frac{\sigma_M(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}da + \frac{\mu_0}{4\pi}\int_V \frac{\rho_M(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3r'$$

Here V denotes the volume of the magnetized object, S its surface (with normal vector $\hat{\mathbf{n}}$), and $\rho_M = -\nabla\cdot M$ and $\sigma_M = M\cdot\hat{\mathbf{n}}$.

Using the expression for our scalar potential

$$V_m = \frac{\mu_0}{4\pi}\frac{\hat{\mathbf{r}}\cdot\mathbf{m}}{r^2} \qquad r = \mathbf{r} - \mathbf{r}'$$

and the total potential is given by

$$V_m = \frac{\mu_0}{4\pi}\int_V \frac{\hat{\mathbf{r}}\cdot\mathbf{M}(\mathbf{r}')}{r^2}d\tau'$$

observing that

$$\nabla'\left(\frac{1}{r}\right) = \frac{\hat{\mathbf{r}}}{r^2}$$

we have

$$V = \frac{\mu_0}{4\pi}\int_V \mathbf{M}\cdot\nabla'\left(\frac{1}{r}\right)d\tau'$$

and integrating by parts and using product rule number 5, gives

$$V = \frac{\mu_0}{4\pi}\left[\int_V \nabla'\cdot\left(\frac{\mathbf{M}}{r}\right)d\tau' - \int_V \frac{1}{r}(\nabla'\cdot\mathbf{M})d\tau'\right]$$

or using the divergence theorem

$$V = \frac{\mu}{4\pi}\oint_S \frac{1}{r}\mathbf{M}\cdot d\mathbf{a}' + \frac{\mu_0}{4\pi}\int_V \frac{1}{r}(-\nabla'\cdot\mathbf{M})d\tau'$$

and we can define

$$\rho_M = -\nabla'\cdot\mathbf{M} \quad \sigma_M = \mathbf{M}\cdot\hat{\mathbf{n}}$$

thus we find

$$\boxed{V_m(\mathbf{r}) = \frac{\mu_0}{4\pi}\oint_S \frac{\sigma_M(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}da + \frac{\mu_0}{4\pi}\int_V \frac{\rho_M(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3r}'$$

The reason why this analogy works so far is that there are, as of yet, no free currents in the systems we're considering. Now we add those currents back in, and we find, of course, that the magnetic field can no longer be given just as the gradient of a potential.

c) Give expressions for the **B** and **H** fields in the presence of both a magnetization **M** and a free current density **J**, still using the magnetic scalar potentials described in parts (a) and (b).

In the presence of free currents we must write the magnetic field as

$$\mathbf{B}(\mathbf{r}) = \boxed{-\nabla V_m + \mu_0 M(\mathbf{r}) + \mu_0\mathbf{H}}$$

where we know that

$$\nabla\times\mathbf{B} = \mu_0\nabla\times\mathbf{M} + \mu_0\nabla\times\mathbf{H} = \mu_0(J_b + J_f) = \mu_0\mathbf{J}$$

and we also know

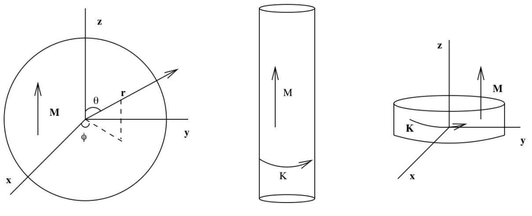
$$\nabla\times\mathbf{H} = J_f$$

this can also be written as

$$\mathbf{J} = \frac{1}{\mu_0}(\nabla\times\mathbf{B}) = J_b + J_f$$

Problem # 7

In solving this problem, keep in mind the boundary conditions for B and H. On a historical note, such cavities were considered in the early days of electromagnetic theory as operational ways of measuring the B and H fields inside magnetic materials.



a) sphere b) needle c) wafer

Suppose the field inside a large piece of magnetic material is $\mathbf{B}_0$, so that $\mathbf{H}_0 = (1/\mu_0)\mathbf{B}_0 - \mathbf{M}$.

a) Now a small spherical cavity is hollowed out of the material. Find the field at the center of the cavity, in terms of $\mathbf{B}_0$ and $\mathbf{M}$. Also find $\mathbf{H}$ at the center of the cavity, in terms of $\mathbf{H}_0$ and $\mathbf{M}$.

We know that the magnetic field for this configuration is given by

$$\mathbf{B} = \mathbf{B}_0 - \mathbf{B}'$$

where **B'** is the magnetic field induced by the magnetization of the material, which was given in example 6.1

$$\mathbf{B}' = \frac{2}{3}\mu_0\mathbf{M}$$

thus the magnetic field is given as

$$\boxed{\mathbf{B} = \mathbf{B}_0 - \frac{2}{3}\mu_0\mathbf{M}}$$

and the **H** field is given by

$$\mathbf{H} = \mathbf{H}_0 - \mathbf{H}'$$

where

$$\mathbf{H}' = \frac{\mathbf{B}'}{\mu_0} - \mathbf{M} = -\frac{1}{3}\mathbf{M}$$

thus we find

$$\boxed{\mathbf{H} = \mathbf{H}_0 + \frac{1}{3}\mathbf{M}}$$

b) Do the same for a long needle-shaped cavity running parallel to **M**.

We know that we cann treat this problem like a solenoid problem, where we have found that the magnetic field is given by

$$\mathbf{B}' = \mu_0 nI = \mu_0 K_b = \mu_0\mathbf{M}$$

and so the magnetic field due to this configuration is given by

$$\mathbf{B} = \mathbf{B}_0 - \mathbf{B}' = \boxed{\mathbf{B}_0 - \mu_0\mathbf{M}}$$

and the auxillary field is given by

$$\mathbf{H} = \mathbf{H}_0 - \mathbf{H}'$$

where

$$\mathbf{H}' = \frac{\mathbf{B}'}{\mu_0} - \mathbf{M} = 0$$

thus

$$\boxed{\mathbf{H} = \mathbf{H}_0}$$

c) Do the same for a thin wafer-shaped cavity perpendicular to **M**.

We know that

$$\mathbf{B} = B_0 - \mathbf{B}'$$

but $\mathbf{B}'$ can be found by assuming this is a loop of wire, which we have solve in problem 1

$$\mathbf{B}' = \frac{\mu_0 K_b h}{2R}\hat{\mathbf{z}}$$

but

$$K_b = \mathbf{M} \times \hat{\mathbf{n}} = M \sin\theta\hat{\phi}$$

where if we take the limit that this thin wafer gets thinner

$$h \to 0 \qquad \mathbf{B}' \to 0$$

thus we find

$$\boxed{\mathbf{B} = B_0}$$

and for the **H** field we find

$$\boxed{\mathbf{H} = H_0 + \mathbf{M}}$$

Chapter 7

Electrodynamics

7.1 Electromotive Force

7.1.1 Ohm's Law

To make a current flow, you have to *push* on the charges. How fast they move, in response to a given push, depends on the nature of the material. For most substances, the current density **J** is proprtional to the *force per unit charge*, **f**

$$\mathbf{J} = \sigma\mathbf{f} \tag{7.1}$$

The proportionality factor σ (not to be confused with the surface charge) is an emperical constant that varies from one material to another; it's called the **conductivity** of the medium. For most purposes metals can be regarded as **perfect conductors** with $\sigma = \infty$.

In principle, the force that drives the charges to produce the current could be anything- chemical, gravitational, or trained ants with tiny harnesses. For our purpose, it's usually an electromagnetic force that does the job. In this case Equation 7.1 becomes

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{7.2}$$

Ordinarily, the velocity of the charges is sufficiently small that the second term can be ignored

$$\mathbf{J} = \sigma\mathbf{E} \tag{7.3}$$

Equation 7.3 is called **Ohm's law**, though the physics behind it is really contained in Equation 7.1, of which 7.3 is just a special case. Ohm's law can also be written as

$$V = IR \tag{7.4}$$

The constant of proportionality R is called the **resistance**; it's a function of the geometry of the arrangement and the conductivity of the medium between two electrodes. Resistance in measured in **ohms** Ω : an ohm is avolt per ampere.

For steady currents and *uniform* conductivity

$$\nabla \cdot \mathbf{E} = \frac{1}{\sigma}\nabla \cdot \mathbf{J} = 0 \tag{7.5}$$

and therefore the charge density is zero; any unbalanced charge resides on the *surface*. It follows, in particular, that Laplace's equation holds within a homogeneous ohmic material carrying a steady current, so all the tools and tricks of Chapter 3 are available for computing the potential. As a results of the collisions between the electrons moving through a medium, the work done by the electric force is converted into heat in the resistor. Since the work done per unit charge is V and the charge flowing per unti time is I, the power delivered is

$$P = VI = I^2 R \tag{7.6}$$

This is called the **Joule heating law**. With I in amperes and R in ohms, P comes out in watts (joules per second).

7.1.2 Electromotive Force

There are *two* forces that drive current around in a circuit: the source $\mathbf{f}_s$, which is ordinarily confined to one portion of the loop (a battery, say), and the *electrostatic* force, which serves to smooth out the flow and communicate the influence of the source to distant parts of the circuit:

$$\mathbf{f} = \mathbf{f}_s + \mathbf{E} \tag{7.7}$$

The physical agency responsible for $\mathbf{f}_s$ can be any one of may different things. Whatever the *mechanism*, its net effect is determined by the line integral of $\mathbf{f}$ around the circuit

$$\mathcal{E} = \oint \mathbf{f} \cdot d\mathbf{l} = \oint \mathbf{f}_s \cdot d\mathbf{l} \tag{7.8}$$

(Because $\oint \mathbf{E} \cdot d\mathbf{l} = 0$ for electrostatic fields, it doesnt matter whether you use **f or** $\mathbf{f}_s$.) $\mathcal{E}$ is called the **electromotive force**, or **emf**, of the circuit. An electromotive force can also arise if a loop of wire is moving through a magnetic field, in this case the **emf** is defined as

$$\mathcal{E} = -\frac{d\Phi}{dt} \tag{7.9}$$

This is known as the **flux rule** for motional ems, where Φ is the flux of **B** through the loop and is defined as

$$\Phi \equiv \int \mathbf{B} \cdot d\mathbf{a} \tag{7.10}$$

Apart from its delightful simplicity, it has the virtue of applying to *nonrectangular* loops moving in *arbitrary* dircetions through *non*uniform magnetic fields; in fact, the loop need not even maintain a fixed shape. The flux rule is a nifty short-cut for calculating motional emf's. It does not contain any new physics. Ocassionally you will run across problems that cannot be handled by the flux rule; for these one must go back to the Lorentz force law itself.

7.2 Electromagnetic Induction

In 1831 Micheal Faraday reported on a series of experiments, including three that can be characterized as follows

Experiment 1. He pulled a loop of wire to the right through a magnetic field. A current flowed in the loop

Experiment 2. He moved the magnet to the left, holding the loop still. Again current flowed in the loop

Experiment 3. With both the loop and the magnet at rest, he changed the *strength* of the field (he used an electromagnet, and varied the current in the coil). Once again, current flowed in the loop

The first experiment, of course, is an example of motional emf, conveniently expressed by the flux rule Equation 7.9. I don't think it will surprise you to learn that exactly the same emf arises in experiment number 2, all that really matters is the *relative* motion of the magnet and the loop. Indeed, in light of special relativity it has to be so. Faraday had an ingenious inspiration

A changing magnetic field induces an electric field

It is this "induced" electric field that accounts for the emf in Experiment 2. Indeed, if (as Faraday found emperically) the emf is again equal to the rate of change of the flux,

$$\mathcal{E} = \oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt} \tag{7.11}$$

then **E** is related to the change in **B** by the equation

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{a} \tag{7.12}$$

This is **Faraday's law**, in integral form,. We can convert it to differential form by applying Stoke's theorem

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{7.13}$$

Note that Faraday's law reduces to the old rule $\oint \mathbf{E} \cdot d\mathbf{l} = 0$ in the static case (constant **B**) as, of course, it should.

In Experiment 3 the magnetic field changes for entirely different reasons, but according to Faraday's law an electrif field will again be induced, giving rise to an emf $-d\Phi/dt$. Indeed, one can subsume all three cases into a kind of **universal flux rule:**

Whenever (and for whatever reason) the magnetic flux through a loop changes, an emf

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

Will appear in the loop.

7.3 Inductance

Suppose you have two loops of wire, at rest. If you run a steady current I_1 around loop 1, it produces a magnetic field $\mathbf{B}_1$. Some of the field lines pass through loop 2; let Φ_2 be the flux of $\mathbf{B}_1$ through 2. You might have a tough time actually *calculating* $\mathbf{B}_1$, but at a glance at the Biot-Savart law

$$\mathbf{B}_1 = \frac{\mu_0}{4\pi} I_1 \oint \frac{d\mathbf{l}_1 \times \hat{\mathbf{r}}'}{r'^2}$$

reveals one significant fact about the field; it is *proportional to the current* I_1. Therefore, so too is the flux through loop 2:

$$\Phi_2 = \int \mathbf{B}_1 \cdot d\mathbf{a}_2 \tag{7.14}$$

Thus

$$\Phi_2 = M_{21} I_1 \tag{7.15}$$

where M_{21} is the constant of proportionality; it is known as the **mutual inductance** of the two loops.

There is a cute formula for the mutual inductance, which can be derived by expressing the flux in terms of the vector potential and invoking Stoke's theorem

$$\Phi_2 = \int \mathbf{B}_1 \cdot d\mathbf{a}_2 = \int (\nabla \times \mathbf{A}_1) \cdot d\mathbf{a}_2 = \oint \mathbf{A}_1 \cdot d\mathbf{l}_2$$

the vector potential can be written as

$$\mathbf{A}_1 = \frac{\mu_0 I_1}{4\pi} \oint \frac{d\mathbf{l}_1}{r'}$$

and hence

$$\Phi_2 = \frac{\mu_0 I_1}{4\pi} \oint \left(\oint \frac{d\mathbf{l}_1}{r'} \right) \cdot d\mathbf{l}_2$$

Evidently

$$M_{21} = \frac{\mu_0}{4\pi} \oint \oint \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{r'} \tag{7.16}$$

This is the **Neumann formula**; it involves a double liune integral. It is not very useful for practical calculations, but it does reveal two important things about mutual inductance:

1. M_{21} is a purely geometrical quantity, having to do with sizes, shapes, and relative positions of the two loops.

2. The integral in Equation 7.15 is unchanged if we switch the roles of loops 1 and 2; it follows that

$$M_{21} = M_{12} = M$$

Suppose now that you *vary* the current in loop 1. The flux through loop 2 will vary accordingly, and Faraday's law says this changing flux will induce an emf in loop 2

$$\mathcal{E}_2 = -\frac{d\Phi_2}{dt} = -M\frac{dI_1}{dt} \tag{7.17}$$

What a remarkable thing: Everytime you change the current in loop 1, and induced current flows in loop 2- even though there are no wires connecting them.

Come to think of it, a changing current not only induces an emf in any nearby loops, it also induces an emf in the source loop $itself$. Once again, the field (and therefore also the flux) is proportional to the current

$$\Phi = LI \tag{7.18}$$

The constant of proportionality L is called the **self-inductance** (or simply the **inductance**) of the loop. As with M, it depends on the geometry (size and shape) of the loop. If the current changes, the emf induced in the loop is

$$\mathcal{E} = -L\frac{dI}{dt} \tag{7.19}$$

Inductance is measured in **henries** (H); a henry is a volt-second per ampere.

7.3.1 Energy in Magnetic Fields

It takes a certain amount of energy to start a current flowing in a circuit. The work done on a unit charge, against the back emf, in one trip around the circuit is $-\mathcal{E}$ (the minus sign records the fact that this is the work done by *you* against the emf, not the work done by the emf. The amount of charge per unit time passing down the wire is I. So the total work done per unit time is

$$\frac{dW}{dt} = -\mathcal{E}I = LI\frac{dI}{dt}$$

If we start with zero current and build it up to a final value I, the work done is

$$\boxed{W = \frac{1}{2}LI^2} \tag{7.20}$$

It does not depend on how long we take to crank up the current, only on the geometry of the loop (in the form of L) and the final current I. We can also write the work done in terms of the magnetic field as

$$\boxed{W = \frac{1}{2\mu_0}\int_{\text{all space}} B^2 d\tau} \tag{7.21}$$

In view of this result, we say the energy is "stored in the magnetic field," in the amount $(B^2/2\mu_0)$ per unit volume.

7.4 Maxwell's Equations

We are now ready to write down all of Maxwell's equations (with corrections), they are summarized as

$$\begin{array}{ccc}
\text{(i)} & \nabla\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0} & \text{(Gauss's law)} \\
\text{(ii)} & \nabla\cdot\mathbf{B} = 0 & \text{(no name)} \\
\text{(iii)} & \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} & \text{(Faraday's law)} \\
\text{(iv)} & \nabla\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\varepsilon_0\frac{\partial\mathbf{E}}{\partial t} & \text{(Ampere's law with Maxwell's correction)}
\end{array} \tag{7.22}$$

Together with the force law

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B}) \tag{7.23}$$

they summarize the entire theoretical content of calssical electrodynamics (save for some special properties of matter). Even the continuity equation

$$\nabla\cdot\mathbf{J} = -\frac{\partial\rho}{\partial t} \tag{7.24}$$

which is the mathematical expression of conservation of charge, can be derived from Maxwell's equations by applying the divergence to number (iv). I have written Maxwell's equation in the traditional way. I think it is logically preferable to write them as

$$\begin{array}{ccc}
\text{(i)} & \nabla\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0} & \text{(Gauss's law)} \\
\text{(ii)} & \nabla\cdot\mathbf{B} = 0 & \text{(no name)} \\
\text{(iii)} & \nabla\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = 0 & \text{(Faraday's law)} \\
\text{(iv)} & \nabla\times\mathbf{B} - \boldsymbol{\mu_0\varepsilon_0}\frac{\boldsymbol{\partial}\mathbf{E}}{\boldsymbol{\partial}\mathbf{t}} = \mu_0\mathbf{J} & \text{(Ampere's law with Maxwell's correction)}
\end{array} \tag{7.25}$$

with the fields ($\mathbf{E}$ and $\mathbf{B}$) on the left hand side and the sources (ρ and $\mathbf{J}$) on the right. This notation emphasizes that all electromagnetic fields are ultimately attributable to charges and currents. Maxwell's equatins tell you how charges produce fields, the force law tells you how fields affect charges.

7.5 Maxwell's Equations in Matter

Maxwell's equations in the form of 7.25 are complete and correct as they stand. However, when you are working with materials that are subject to electric and magnetic polarization there is a more convenient way to write them. In terms of $free$ charge and currents they are summarized as follows

$$\begin{aligned} \text{(i)}\ \nabla\cdot\mathbf{D} &= \rho_f \quad \text{(iii)}\ \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \\ \text{(ii)}\ \nabla\cdot\mathbf{B} &= 0 \quad \text{(iv)}\ \nabla\times\mathbf{H} = \mathbf{J}_f + \frac{\partial\mathbf{D}}{\partial t} \end{aligned} \tag{7.26}$$

Some people regard these as the "true" Maxwell's equations, but they are in no way more "general" than 7.25; they simply reflect a convenient division of charge and current into free and non-free parts. And they have the disadvantage of hybrid notation, since they contain both **E** and **D**, both **B** and **H**. They must be supplemented, therefore, by appropriate **constitutive relations**, giving **D** and **Hb** in terms of **E** and **B**. These depend on the nature of the material; for linear media

$$\mathbf{P} = \varepsilon_0\chi_e\mathbf{E} \quad \text{and} \quad \mathbf{M} = \chi_m\mathbf{H}$$

so

$$\mathbf{D} = \varepsilon\mathbf{E} \quad \text{and} \quad \mathbf{H} = \frac{\mathbf{B}}{\mu}$$

where $\varepsilon = \varepsilon_0(1+\chi_e)$ and $\mu = \mu_0(1+\chi_m)$. Incidentally, you'll remember that **D** is called the electric "displacement"; thats why the second term in the Ampere/Maxwell equation (iv) is called the **displacement current.**

7.6 Boundary Conditions

In general, the fields **E,B,D,** and **H** will be discontinuous at a boundary between two different media, or at a surface that carries charge density σ or current density **K**. The explicit form of these discontinuities can be deduced from Maxwell's equations 7.27, in their integral form

$$\begin{aligned} \text{(i)}\ \oint_S \mathbf{D}\cdot d\mathbf{a} &= Q_{fenc} \quad \text{(iii)}\ \oint_{\mathcal{P}} \mathbf{E}\cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{a} \\ \text{(ii)}\ \oint_S \mathbf{B}\cdot d\mathbf{a} &= 0 \quad \text{(iv)}\ \oint_{\mathcal{P}} \mathbf{H}\cdot d\mathbf{l} = I_{fenc} + \frac{d}{dt}\int_S \mathbf{D}\cdot d\mathbf{a} \end{aligned} \tag{7.27}$$

where (i) and (ii) are defined over any closed surface S, and (iii) and (iv) are for any surface S bounded by the closed loop $\mathcal{P}$. For (i) we find

$$\boxed{D_1^{\perp} - D_2^{\perp} = \sigma_f} \tag{7.28}$$

for (ii) we find

$$\boxed{B_1^{\perp} - B_2^{\perp} = 0} \tag{7.29}$$

for (iii) we find

$$\boxed{\mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = 0} \tag{7.30}$$

and for (iv) we find

$$\boxed{\mathbf{H}_1^{\parallel} - \mathbf{H}_2^{\parallel} = \mathbf{K}_f \times \hat{\mathbf{n}}} \tag{7.31}$$

Equation 7.29-7.32 are the general boundary conditions for electrodynamics. In the case of *linear* media, they can be expressed in terms of **E** and **B** alone

$$\begin{aligned} \text{(i)}\ \varepsilon_1 E_1^{\perp} - \varepsilon_2 E_2^{\perp} &= \sigma_f \quad \text{(iii)}\ \mathbf{E}_1^{\|} - \mathbf{E}_2^{\|} = 0 \\ \text{(ii)}\ B_1^{\perp} - B_2^{\perp} = 0 &= 0 \quad \text{(iv)}\ \frac{\mathbf{1}}{\mu_{\mathbf{1}}}\mathbf{H}_1^{\|} - \frac{1}{\mu_2}\mathbf{H}_2^{\|} = \mathbf{K}_f \times \hat{\mathbf{n}} \end{aligned} \tag{7.32}$$

In particular, if there is no free charge or free current at the interface, then

$$\begin{aligned} \text{(i)}\ \varepsilon_1 E_1^{\perp} - \varepsilon_2 E_2^{\perp} &= 0 \quad \text{(iii)}\ \mathbf{E}_1^{\|} - \mathbf{E}_2^{\|} = 0 \\ \text{(ii)}\ B_1^{\perp} - B_2^{\perp} = 0 &= 0 \quad \text{(iv)}\ \frac{\mathbf{1}}{\mu_{\mathbf{1}}}\mathbf{H}_1^{\|} - \frac{1}{\mu_2}\mathbf{H}_2^{\|} = 0 \end{aligned} \tag{7.33}$$

7.7 Problems and Solutions

Problem # 1

Notice the following parallel:

$$\begin{aligned} \nabla \times \mathbf{D} &= 0 \quad \nabla \times \mathbf{E} = 0 \quad \varepsilon_0 \mathbf{E} = \mathbf{D} - \mathbf{P} \\ \nabla \times \mathbf{B} &= 0 \quad \nabla \times \mathbf{H} = 0 \quad \varepsilon_0 \mathbf{H} = \mathbf{B} - \mu_0 \mathbf{M} \end{aligned}$$

thus, the transcription $\mathbf{D} \rightarrow \mathbf{B}, \mathbf{E} \rightarrow \mathbf{H}, \mathbf{P} \rightarrow \mu_0 \mathbf{M}, \varepsilon_0 \rightarrow \mu_0$ turns an electrostatic problem into an analogous magnetostatic one. Use this observation together with your knowledge of the electrostatic results, to rederive

a) The magnetic field inside a uniformly magnetized sphere. (Eq. 6.16)

The magnetic field of a uniformly magnetized sphere is given by Equatio 6.16

$$\mathbf{B} = \frac{2}{3}\mu_0 \mathbf{M}$$

The electric field of a uniformly polarized sphere is given by Equation 4.14

$$\mathbf{E} = -\frac{1}{3\varepsilon_0}\mathbf{P}$$

if we let

$$\mathbf{E} \rightarrow \mathbf{H} \quad \mathbf{P} \rightarrow \mu_0 \mathbf{M} \quad \varepsilon_0 \rightarrow \mu_0 \quad \mu_0 \mathbf{H} = \mathbf{B} - \mu_0 \mathbf{M}$$

we find

$$\mathbf{H} = -\frac{1}{3}\mathbf{M} \quad \mu_0 \mathbf{H} = -\frac{1}{3}\mu_0 \mathbf{M}$$

which simplifies to

$$\mathbf{B} - \mu_0\mathbf{M} = -\frac{1}{3}\mu_0\mathbf{M}$$

thus

$$\boxed{\mathbf{B} = \frac{2}{3}\mu_0\mathbf{M}}$$

b) The magnetic field inside a sphere of linear magnetic material in an otherwise uniform magnetic field. (Prob. 6.18)

From example 4.7 we know that the electric field inside a linear dialectric sphere is given by Equation 4.49

$$\mathbf{E} = \frac{3}{2+\varepsilon_r}\mathbf{E}_0$$

If we let

$$\mathbf{E} \rightarrow \mathbf{H} \quad \varepsilon_r = 1+\chi_e \rightarrow \varepsilon_m = 1+\chi_m \quad \mathbf{E}_0 \rightarrow \mathbf{H}_0$$

thus we have

$$\mathbf{H} = \frac{3}{2+\chi_m}\mathbf{H}_0$$

if we let

$$\mathbf{P} = \varepsilon_0\chi_e\mathbf{E} = \mu_0\chi_m\mathbf{H} = \mu_{\mathbf{0}}\mathbf{M}$$

therefore

$$\mathbf{M} = \chi_m\mathbf{H}$$

we also know

$$\mu_0\mathbf{H} = \mathbf{B} - \mu_0\mathbf{M} \quad \rightarrow \mathbf{H} = \frac{\mathbf{B}}{\mu_0(1+\chi_m)}$$

and now to find what $\mathbf{H}_0$ is we must use

$$\mu_0\mathbf{H}_0 = \mathbf{B}_0 - \mu_0\mathbf{M} = \mathbf{B}_0 \quad \mathbf{M} = 0 \ \ \text{outside}$$

thus

$$\mathbf{H}_0 = \frac{\mathbf{B}_0}{\mu_0}$$

and so we find that the magnetic field inside a sphere of linear magnetic material in an otherwise uniform magnetic field is given by

$$\boxed{\mathbf{B} = \frac{3(1+\chi_m)}{3+\chi_m}\mathbf{B}_0}$$

c) The average magnetic field over a sphere, due to steady currents within the sphere. (Eq. 5.89)

We know that the average magnetic fields over a sphere due to steady currents is given by Equation 5.89

$$\mathbf{B}_{\text{ave}} = \frac{\mu_0}{4\pi}\frac{2\mathbf{m}}{R^3}$$

we can derive this using our knowledge of what the average electric field due to a similar electrostatic problem. The average electric field is given by Equation 3.105 as

$$\mathbf{E}_{\text{ave}} = -\frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}}{R^3}$$

we also know that

$$\mathbf{P} = \frac{\mathbf{p}}{V}$$

where we know that the volume is given by

$$V = \frac{4}{3}\pi R^3$$

thus we can rewrite the average electric field as

$$\mathbf{E}_{\text{ave}} = -\frac{1}{3\varepsilon_0}\mathbf{P}$$

which is identical to part a) of this problem which yielded

$$\mathbf{B}_{\text{ave}} = \frac{2}{3}\mu_0\mathbf{M}$$

but we know that this can also be written as

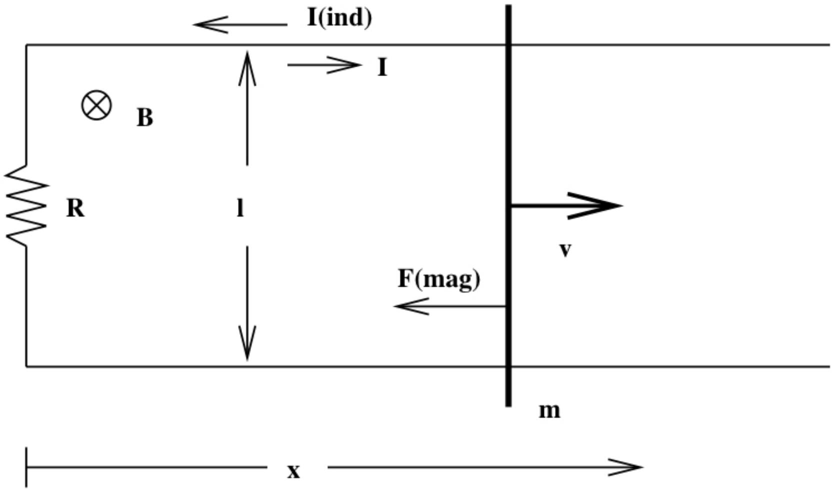
$$\mathbf{M} = \frac{\mathbf{m}}{V} \quad \Rightarrow \mathbf{B}_{\text{ave}} = \frac{2}{3V}\mu_0\mathbf{m}$$

where this can be written as

$$\boxed{\mathbf{B}_{\text{ave}} = \frac{2\mu_0}{4\pi}\frac{\mathbf{m}}{R^3}}$$

Problem # 2

A metal bar of mass *m* slides frictionless on two parallel conducting rails a distance *l* apart (Figure 1). A resistor *R* is connected across the rails and a uniform magnetic field **B**, pointing into the page, fills the entire region.



a) If the bar moves to the right at speed *v*, what is the current in the resistor? In what direction does it flow?

Since we know that

$$I = \frac{V}{R}$$

from Ohm's law, where V is now the induced emf, which is given by

$$V = \mathcal{E} = -\frac{d\Phi}{dt}$$

and Φ is the magnetic flux which is given by

$$\Phi = \int \mathbf{B} \cdot d\mathbf{a} = \mathbf{B}xl$$

where the normal are vector is into the page. The emf is given by

$$\mathcal{E} = -\frac{dx}{dt}Bl = -Blv$$

thus the current is given by

$$\boxed{I = -\frac{Blv}{R} \qquad \text{counter-clockwise}}$$

b) What is the magnetic force on the bar? In what direction?

We know that the Lorentz force is given by

$$\mathbf{F}_B = q(v \times \mathbf{B}) = qv\mathbf{B}$$

but since we know that

$$qv = Il$$

we find

$$\mathbf{F}_B = Il\mathbf{B} = -\frac{vl^2B^2}{R}(\hat{\mathbf{x}})$$

there is a magnetic force in the $-\hat{\mathbf{x}}$ direction.

c) If the bar starts out with speed v_0 at time $t = 0$, and is left to slide, what is its speed at a later time t?

Since we know that

$$\mathbf{F}_B = ma = m\frac{dv}{dt}$$

we can solve for the velocity, i.e

$$-\frac{vl^2B^2}{R} = m\frac{dv}{dt}$$

which can be written in a more suggestive way

$$-\int_0^t \frac{l^2B^2}{Rm}dt = \int_{v_0}^{v} \frac{dv}{v}$$

which is a simple integral with a solution of

$$\boxed{v = v_0 e^{-kt} \qquad k = \frac{l^2B^2}{Rm}}$$

d) The initial kinetic energy of the bar was, of course, $\frac{1}{2}mv_0^2$. Check that the energy delivered to the resistor is exactly $\frac{1}{2}mv_0^2$.

We know that the energy is given by

$$E = \int P dt$$

where P is the power that is obtained from the loss of kinetic energy of the bar, the power is defined as

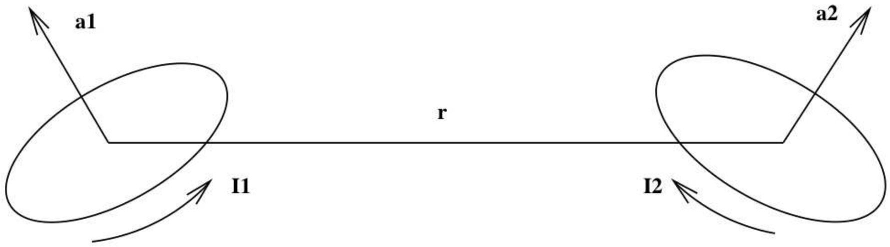
$$P = I^2 R = \frac{l^2 v^2 B^2}{R} = \frac{l^2 B^2}{R} v_0^2 e^{-2kt}$$

where k has been previously defined, so we find that the total energy delivered to this transistor is

$$E = \frac{l^2 B^2}{R} \int_0^\infty e^{-2kt} dt = \frac{l^2 B^2}{R} \frac{Rm}{2l^2 B^2} v_0^2 = \frac{1}{2} m v_0^2$$

Problem # 3

Two tiny wire loops, with areas $\mathbf{a}_1$ and $\mathbf{a}_2$, are situated a displacement $\mathbf{r}$ apart (Figure 2).



a) Find their mutual inductance. [*Hint* : Treat them as magnetic dipoles, and use Eq. 5.87.] Is your formula consistent with Eq. 7.23?

Using Equation 5.87

$$\mathbf{B}_{\text{dip}} = \frac{\mu_0}{4\pi} \frac{1}{r^3} [3(\mathbf{m} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}]$$

using Biot-Savart law and the definition of the flux thru loop 2 from the magnetic field due to loop 1

$$\Phi_2 = \int \mathbf{B}_1 \cdot d\mathbf{a}_2$$

we can define the mutual inductance of the two loops as

$$\Phi_2 = M_{21} I_1$$

where M_{21} is the **mutual inductance**. Using this we can now find

$$M_{21} = \frac{\Phi_2}{I_1} = \frac{1}{I_1} \int \mathbf{B}_1 \cdot d\mathbf{a}_2$$

given the definition of the magnetic field due to a dipole we find

$$M_{21} = \frac{1}{I_1} \frac{\mu_0}{4\pi} \frac{1}{r^3} [3(\mathbf{m}_1 \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}_1] \cdot \mathbf{a}_2$$

knowing

$$\mathbf{m}_1 = \mathbf{a}_1 I_1 \quad \mathbf{m}_2 = \mathbf{a}_2 I_2$$

gives us

$$M_{21} = \frac{\mu_0}{4\pi}\frac{1}{r^3}[3(\mathbf{a}_1 \cdot \hat{\mathbf{r}})(\mathbf{a}_2 \cdot \hat{\mathbf{r}}) - \mathbf{a}_1 \cdot \mathbf{a}_2]$$

we can also show that the is the same if we look at it from the perspective of the other loop,

$$M_{12} = \frac{\Phi_1}{I_2} = \frac{1}{I_2}\int \mathbf{B}_2 \cdot d\mathbf{a}_1$$

given the definition of the magnetic field due to a dipole we find

$$M_{12} = \frac{1}{I_2}\frac{\mu_0}{4\pi}\frac{1}{r^3}[3(\mathbf{m}_2 \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}_2] \cdot \mathbf{a}_1$$

knowing

$$\mathbf{m}_1 = \mathbf{a}_1 I_1 \quad \mathbf{m}_2 = \mathbf{a}_2 I_2$$

gives us

$$M_{21} = \frac{\mu_0}{4\pi}\frac{1}{r^3}[3(\mathbf{a}_2 \cdot \hat{\mathbf{r}})(\mathbf{a}_1 \cdot \hat{\mathbf{r}}) - \mathbf{a}_2 \cdot \mathbf{a}_1]$$

which is excatly the same, thus we can conclude

$$M_{21} = M_{12} = M$$

b) Suposse a current I_1 is flowing in loop 1, and we propose to turn on a current I_2 in loop 2. How much work must be done, against the mutually induced emf, to keep the current I_1 flowing in loop 1? In light of this result, comment on Eq.6.35.

Since we know that the total work done per unit time is given by

$$\frac{dW}{dt} = -\mathcal{E} I$$

where $\mathcal{E}$ is the induced emf and I is the current going thru the loop. We need to find

$$dW = -\mathcal{E}_1 I_1 dt$$

but we know that the induced emf on 1 is given by

$$\mathcal{E}_1 = -\frac{d\Phi_1}{dt} \quad \Phi_1 = \int \mathbf{B}_2 \cdot d\mathbf{a}_1 = M I_2$$

and so we find

$$\mathcal{E}_1 = -M\frac{dI_2}{dt}$$

putting this into our equation for the work yields

$$dW = M I_1 dI_2 \quad \Rightarrow W = M I_1 I_2$$

We found the mutual inductance from part a), substituting this in gives us

$$W = \frac{\mu_0}{4\pi}\frac{1}{r^3}[3(\mathbf{a}_1 I_1 \cdot \hat{\mathbf{r}})(\mathbf{a}_2 I_2 \cdot \hat{\mathbf{r}}) - \mathbf{a}_1 I_1 \cdot \mathbf{a}_2 I_2]$$

where we have defined

$$\mathbf{m}_1 = \mathbf{a}_1 I_1 \quad \mathbf{m}_2 = \mathbf{a}_2 I_2$$

which gives us

$$\boxed{W = \frac{\mu_0}{4\pi}\frac{1}{r^3}[3(\mathbf{m}_1\cdot\hat{\mathbf{r}})(\mathbf{m}_2\cdot\hat{\mathbf{r}}) - \mathbf{m}_1\cdot\mathbf{m}_2]}$$

this looks very similar to the interaction energy of two magnetic dipoles.

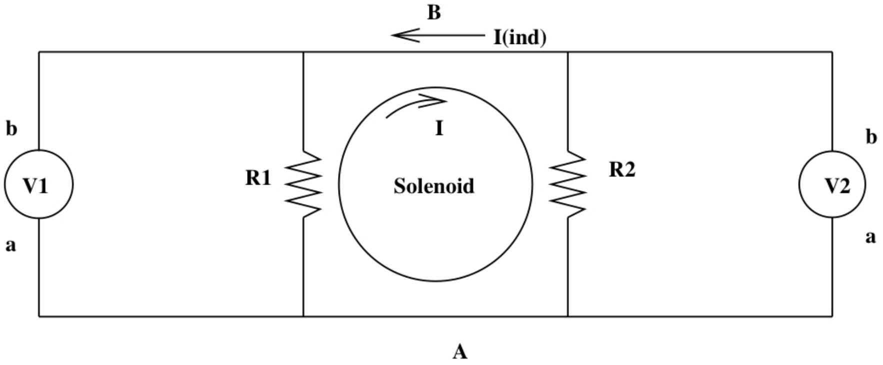
$$W = -U$$

where U is the interaction energy of two magnetic dipoles

$$U = \frac{\mu_0}{4\pi}\frac{1}{r^3}[\mathbf{m}_1\cdot\mathbf{m}_2 - 3(\mathbf{m}_1\cdot\hat{\mathbf{r}})(\mathbf{m}_2\cdot\hat{\mathbf{r}})]$$

Problem # 4

The current in a long solenoid is increasingly linear in time, so that the flux is proportional to $t:\Phi = \alpha t$. Two voltmeters are connected to diametrically opposite points (A and B), together with resistors (R_1 and R_2), as shown in Figure 3. What is the reading at each voltmeter? Assume that these are *ideal* voltmeters that draw negligable current (they have huge internal resistance), and that a voltmeter registers $\int_a^b \mathbf{E}\cdot d\mathbf{l}$ between the terminals and through the meter. [*Answer* : $V_1 = \alpha R_1/(R_1+R_2)$; $V_2 = -\alpha R_2/(R_1+R_2)$. Notice that $V_1 \neq V_2$, even though they are connected to the same points!]



Since we know that these resistors are in series we can write the current as

$$I = \frac{V_{ind}}{R_T} = \frac{V_{ind}}{R_1+R_2}$$

but we know that the induced emf is given by

$$V_{ind} = \mathcal{E} = -\frac{d\Phi}{dt} = -\alpha$$

thus the current running thru this loop is given by

$$I = -\frac{\alpha}{R_1+R_2}$$

We can find what the potential is across resistor 1 using

$$V_1 = -\int_b^a \mathbf{E}\cdot d\mathbf{l} = -IR_1 = \frac{\alpha R_1}{R_1+R_2}$$

and to find the potential across resistor 2 we use

$$V_2 = \int_a^b \mathbf{E} \cdot d\mathbf{l} = IR_2 = -\frac{\alpha R_2}{R_1 + R_2}$$

thus

$$\boxed{V_1 = \frac{\alpha R_1}{R_1 + R_2} \qquad V_2 = -\frac{\alpha R_2}{R_1 + R_2}}$$

Problem # 5

Suppose

$$\mathbf{E}(\mathbf{r},t) = -\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\theta(vt - r)\hat{\mathbf{r}};\ \mathbf{B}(\mathbf{r},t) = 0$$

(the theta function is defined in Prob. 1.45b). Show that these fields satisfy all of Maxwell's equations, and determine ρ and **J**. Describe the physical situation that give rise to these fields.

If we look at

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

Using product rule # 5

$$\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$$

where $f = \theta(vt - r)$ and $\mathbf{A} = \frac{\hat{\mathbf{r}}}{r^2}$, thus we find

$$\nabla \cdot \mathbf{E} = -\frac{q}{4\pi\varepsilon_0}\left[\theta(vt - r)\left(\nabla \cdot \frac{\hat{\mathbf{r}}}{r^2}\right) + \frac{\hat{\mathbf{r}}}{r^2} \cdot (\nabla(\theta(vt - r))\right] = \frac{\rho}{\varepsilon_0}$$

Equation 1.99 in Griffith's gives

$$\nabla \cdot \frac{\hat{\mathbf{r}}}{r^2} = 4\pi\delta^3(\mathbf{r})$$

and

$$\frac{\hat{\mathbf{r}}}{r^2} \cdot (\nabla(\theta(vt - r)) = \frac{\hat{\mathbf{r}}}{r^2} \cdot \left(\frac{\partial}{\partial r}(\theta(vt - r)\right) = \frac{\hat{\mathbf{r}}}{r^2} \cdot \left(\frac{d\theta}{d(vt - r)}\frac{d(vt - r)}{dr}\right) = -\frac{\delta(vt - r)}{r^2}$$

Since we know that

$$\frac{d\theta}{d(vt - r)} = \delta(vt - r) \quad \frac{d(vt - r)}{dr} = -1$$

Thus we find that

$$\boxed{\rho = -q\left[\theta(vt - r)\delta^3(\mathbf{r}) - \frac{\delta(vt - r)}{4\pi r^2}\right]}$$

The second Maxwell equation is

$$\boxed{\nabla \cdot \mathbf{B} = 0 \quad \mathbf{B} = 0}$$

The third equation is

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Since $\mathbf{B} = 0$, we find

$$\boxed{\nabla \times \mathbf{E} = 0}$$

For the last Maxwell equation

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

We know that

$$\nabla \times \mathbf{B} = 0$$

Thus

$$\mathbf{J} = -\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

We can find

$$\frac{\partial \mathbf{E}}{\partial t} = -\frac{q}{4\pi\varepsilon_0 r^2}\frac{\partial}{\partial t}\theta(vt-r) = -\frac{q}{4\pi r^2}\left[\frac{d\theta}{d(vt-r)}\frac{d(vt-r)}{dt}\right] = -\frac{q}{4\pi r^2} v\delta(vt-r)$$

Thus

$$\boxed{\mathbf{J} = \frac{q}{4\pi r^2} v\delta(vt-r)}$$

Problem # 6

Assuming that "Coulomb's law" for magnetic charges (q_m) reads

$$\mathbf{F} = \frac{\mu_0}{4\pi}\frac{q_{m_1} q_{m_2}}{r'}\hat{\mathbf{r}}'$$

where r' is Griffith's script r, which is the seperation vector. Work out the force law for a monopole q_m moving with velocity $\mathbf{v}$ through electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$.

Looking at the paper written in 1961 by Robert Katz from Kansas state university we find that Maxwell's equations can be written as

$$\begin{aligned} \nabla \cdot \mathbf{D} &= \rho \quad \nabla \times \mathbf{E} = -\left(\frac{\partial \mathbf{B}}{\partial t} + \mathbf{J}_m\right) \\ \nabla \cdot \mathbf{B} &= \rho_m \quad \nabla \times \mathbf{H} = \left(\frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}\right) \end{aligned}$$

he has defined the following properties

$$\mathbf{D} = \frac{q}{4\pi r^2}\hat{\mathbf{r}} \quad \mathbf{B} = \frac{q_m}{4\pi r^2}\hat{\mathbf{r}} \quad \mathbf{H} = \frac{(q\mathbf{v} \times \hat{\mathbf{r}})}{4\pi r^2} \quad \mathbf{E} = -\frac{(q_m \mathbf{v} \times \hat{\mathbf{r}})}{4\pi r^2}$$

set up in this way magnetic poles satisfy the continuity equation; we demend the conservation of poles. We can immideatly write the magnetic analog of the Lorentz force equation as

$$\mathbf{F}_m = q_m[\mathbf{H} - \mathbf{v} \times \mathbf{D}] = q_m(\mathbf{B} - \mathbf{v} \times \mathbf{B}) = q_m(\mathbf{B} - \frac{1}{c^2}\mathbf{v} \times \mathbf{E})$$

where the $1/c^2$ was added in order to make the units work out, and also due to the motivation that $E = cB$. We have shown why these are the units needed in the next part of this explanation.

We claim to find this using simple unit analysis and Equations 7.43 from Griffith's. We know that the Lorentz force for electric charges is given by

$$\mathbf{F}_e = q_e[\mathbf{E} + \mathbf{v} \times \mathbf{B}]$$

Thus by analogy we should find

$$\mathbf{F}_m = q_m[\mathbf{B} \pm c_1(\mathbf{v} \times \mathbf{E})]$$

we need to find a constant c_1 that would make the units work out. What are the units of q_m? We know that the units of the force must be in Newtons and we know that the units of the magnetic field are

$$\mathbf{B} \to \frac{\mathrm{N}}{\mathrm{Am}} \quad \mathbf{F}_m = q_m\mathbf{B}$$

where the units of q_m must be (Am). Now looking at the second term we find

$$\mathbf{F}_m = q_m c_1(\mathbf{v} \times \mathbf{E})$$

If

$$q_m \to \mathrm{Am} \quad \mathbf{v} \times \mathbf{E} \to \frac{\mathrm{Nm}}{\mathrm{As}^2}$$

Thus the units of c_1 must be

$$c_1 \to \frac{\mathrm{s}^2}{\mathrm{m}^2}$$

In order to have the force be in Newtons, a good choice for this units would be the speed of light c, thus

$$\mathbf{F}_m = q_m[\mathbf{B} \pm \frac{1}{c^2}\mathbf{v} \times \mathbf{E}]$$

now for the sign we must look at Equations 7.43 in Griffith's, we can see that

$$\nabla \times \mathbf{E} = -\mu_0\mathbf{J}_m - \frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{B} = \mu_0\mathbf{J}_e + \mu_0\varepsilon_0\frac{\partial \mathbf{E}}{\partial t}$$

we can see that when $\nabla \times \mathbf{B} \to \nabla \times \mathbf{E}$ we gain a minus sign and thus we can conclude that

$$\boxed{\mathbf{F}_m = q_m[\mathbf{B} - \frac{1}{c^2}\mathbf{v} \times \mathbf{E}]}$$

Problem # 7

In a **perfect conductor**, the conductivity is infinite, so **E**=0 (for $\mathbf{J} = \sigma\mathbf{E}$), and any net charge resides on the surface (just as it does for an *imperfect* conductor, in *electrostatics*).

a) Show that the magnetic field is constant ($\partial\mathbf{B}/\partial t = 0$), inside a perfect conductor.

We know that

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Since we know that $\mathbf{E} = 0$ we find

$$\frac{\partial \mathbf{B}}{\partial t} = 0$$

This will only be true if the magnetic field is constant.

b) Show that the magnetic flux through a perfectly conducting loop is contant.

We know that

$$\mathcal{E} = -\int \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt} = 0$$

Because $\mathbf{E} = 0$, and this will only be true if Φ is a constant.

A **superconductor** is a perfect conductor with the *additional* property that the (constant) $\mathbf{B}$ inside is in fact *zero*. (This “flux exclusion” is known as the **Meissner effect**.)

c) Show that the current in a superconductor is confined to the surface.

Using Maxwell’s equation

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} = 0$$

and since we know that $\mathbf{E} = 0$ than

$$\mathbf{J} = 0$$

and any charges that are moving must reside on the surface.

d) Superconductivity is lost above a certain critical temperature (T_c), which varies from one material to another. Suppose you had a sphere (radius a) above its critical temperature, and you held it in a uniform magnetic field $B_0\hat{\mathbf{z}}$ while cooling it below T_c. Find the induced surface current density $\mathbf{K}$, as a function of the polar angle θ.

Using Example 5.11 in Griffith’s we find that the magnetic field inside of a rotating magnetized sphere is given by

$$\mathbf{B} = \frac{2}{3}\mu_0 \sigma r \omega \hat{\mathbf{z}}$$

This is the electric field before the temperature falls below the critical temperature $T > T_c$. Once we decrease the temperature below the critical temperature we know that the magnetic field inside the sphere must be zero due to the “Meissner effect” and thus we know that we must induce a surface current that will oppose the external field. Thus

$$\mathbf{B}_{external} = B_0\hat{\mathbf{z}} \quad \mathbf{B}_{induced} = -B_0\hat{\mathbf{z}}$$

Thus we know that

$$\mathbf{B} = \frac{2}{3}\mu_0 \sigma r \omega \hat{\mathbf{z}} = -B_0\hat{\mathbf{z}} \quad \sigma r \omega = -\frac{3}{2}\frac{B_0}{\mu_0}$$

and since we know that the surface current is defined as

$$\mathbf{K} = \sigma(\omega \times \mathbf{r}) = \sigma r \omega \sin\theta \hat{\phi}$$

and using the result above we find

$$\boxed{\mathbf{K} = -\frac{3}{2}\frac{B_0}{\mu_0}\sin\theta\hat{\phi}}$$

Problem # 8

A familiar demonstration of superconductivity is the levitation of a magnet over a piece of superconducting material. This phenomenom can be analyzed using the method of images. Treat the magnet as a perfect dipole $\mathbf{m}$, a height z above the origin (and constrained to point in the z direction), and pretend that the superconductor occupies the entire half-space below the xy plane. Because of the Meissner effect, $\mathbf{B} = 0$ for $z \leq 0$, and since $\mathbf{B}$ is divergenceless, the normal (z) component is continuous, so $B_z = 0$ just *above* the surface. This boundary condition is met by the image configuration in which an identical dipole is placed at $-z$, as a stand-in for the superconductor; the two arrangements therefore produce the same magnetic field in the region $z > 0$.

a) Which way should the magnetic dipole point ($+z$ or $-z$)?

We know that the image dipole must point in the $-z$ direction so that at $z = 0$ the perpendicular component of the magnetic field must be 0. Mathematically

$$\mathbf{m}_1 = m_1\hat{\mathbf{z}} \quad \mathbf{m}_2 = -m_2\hat{\mathbf{z}}$$

where $\mathbf{m}_1$ is the original dipole and $\mathbf{m}_2$ is the image dipole.

b) Find the force on the magnet due to the induced currents in the superconductor (which is to say, the force due to the image dipole). Set it equal to M_g (where M is the mass of the magnet) to determine the height h at which the magnet will "float." [$Hint$: refer to Prob 6.3]

We know that the force on a dipole due to a magnetic field is defined as

$$\mathbf{F} = \nabla(\mathbf{m}\cdot\mathbf{B})$$

since this is an image problem we know that this can also be written as

$$\mathbf{F} = \nabla(\mathbf{m}_1\cdot\mathbf{B_2})$$

which says that there is a force on the original dipole to to the magnetic field from the image dipole, which is defined as

$$\mathbf{B}_2 = \frac{\mu_0}{4\pi r^3}[3(\mathbf{m}_2\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}_2]$$

we know that

$$\mathbf{m}_1 = m_1\hat{\mathbf{z}} \quad \mathbf{m}_2 = -m_1\hat{\mathbf{z}} \quad \hat{\mathbf{r}} = \hat{\mathbf{z}}$$

we can now write the magnetic field from the image dipole as

$$\mathbf{B}_2 = \frac{\mu_0}{4\pi r^3}[-3(m_1\hat{\mathbf{z}}\cdot\hat{\mathbf{z}})\hat{\mathbf{z}} + m_1\hat{\mathbf{z}}] = -\frac{\mu_0}{4\pi r^3}[2m_1\hat{\mathbf{z}}]$$

Thus we find the force to be given as

$$\mathbf{F} = -\frac{\mu_0}{2\pi}\nabla\left(\frac{m_1^2}{r^3}\right)$$

Thus we find

$$\mathbf{F} = \frac{3\mu_0 m_1^2}{2\pi r^4}$$

we know that

$$r = 2z$$

which is just the speration distance between the two dipoles, thus the force is given by

$$\mathbf{F} = \frac{3\mu_0 m_1^2}{32\pi z^4} = M_g$$

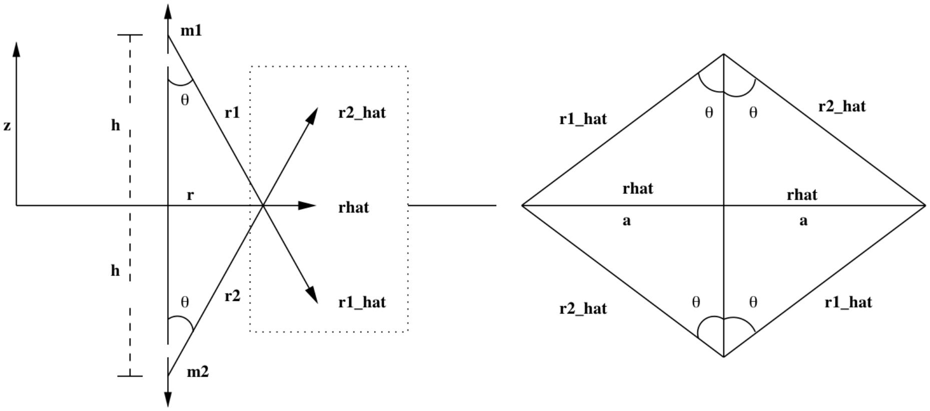
If we let $z = h$ we find that the height is found to be

$$\boxed{h = \left(\frac{3\mu_0 m_1^2}{32\pi M_g}\right)^{1/4}}$$

c) The induced current on the surface of the superconductor (the *xy* plane) can be determined from the boundary condition on the *tangential* component of **B** ($\mathbf{B} = \mu_0(\mathbf{K}\times\hat{\mathbf{z}})$. Using the field you get from the image configuration, Show that

$$\mathbf{K} = -\frac{3mrh}{2\pi(r^2+h^2)^{5/2}}\hat{\phi}$$

where r is the distance from the origin.



We can find what the induced current if we know what the magnetic field is on the *xy* plane, thus we have to find what the total magnetic field is at $z = 0$ this can be achieved by figuring out what the magnetic field is due to the two dipoles, we know that the magnetic field from a dipole is given by

$$\mathbf{B}_{\text{dip}} = \frac{\mu_0}{4\pi r^3}[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}]$$

thus for dipole 1 which I will consider my real dipole we find

$$\mathbf{B}_1 = \frac{\mu_0}{4\pi r_1^3}[3(\mathbf{m}_1\cdot\hat{\mathbf{r}_1})\hat{\mathbf{r}_1} - \mathbf{m}_1]$$

and like wise for the image dipole

$$\mathbf{B}_2 = \frac{\mu_0}{4\pi r_2^3}[3(\mathbf{m}_2\cdot\hat{\mathbf{r}_2})\hat{\mathbf{r}_2} - \mathbf{m}_2]$$

thus we know that the total magnetic field is given by

$$\mathbf{B}_T = \mathbf{B}_1 + \mathbf{B}_2 = \frac{\mu_0}{4\pi}\left[\frac{1}{r_1^3}[3(\mathbf{m}_1\cdot\hat{\mathbf{r}_1})\hat{\mathbf{r}_1} - \mathbf{m}_1] + \frac{1}{r_2^3}[3(\mathbf{m}_2\cdot\hat{\mathbf{r}_2})\hat{\mathbf{r}_2} - \mathbf{m}_2]\right]$$

since we know that

$$r_1 = r_2 \quad \mathbf{m}_2 = -\mathbf{m}_1 = -m_1\hat{\mathbf{z}}$$

thus we can write the above expression as

$$\mathbf{B}_T = \frac{\mu_0 3}{4\pi r_1^3}[(\mathbf{m}_1\cdot\hat{\mathbf{r}_1})\hat{\mathbf{r}_1} - (\hat{\mathbf{m}}_1\cdot\hat{\mathbf{r}_2})\hat{\mathbf{r}_2}]$$

or we can also write this as

$$\mathbf{B}_T = \frac{\mu_0 3m_1}{4\pi r_1^3}[(\hat{\mathbf{z}}\cdot\hat{\mathbf{r}_1})\hat{\mathbf{r}_1} - (\hat{\mathbf{z}}\cdot\hat{\mathbf{r}_2})\hat{\mathbf{r}_2}]$$

we can see from the diagram that

$$\mathbf{m}_1 \cdot \hat{\mathbf{r}_1} = \sin\theta\hat{\mathbf{r}} - \cos\theta\hat{\mathbf{z}} \quad \mathbf{m}_1 \cdot \hat{\mathbf{r}_2} = \sin\theta\hat{\mathbf{r}} + \cos\theta\hat{\mathbf{z}}$$

or from the diagram on the right we also find

$$\hat{\mathbf{z}} \cdot \hat{\mathbf{r}_1} = -\cos\theta \quad \hat{\mathbf{z}} \cdot \hat{\mathbf{r}_2} = \cos\theta$$

thus the total magnetic field can now be written as

$$\mathbf{B}_T = -\frac{3\mu_0 m_1}{4\pi r_1^3}[\cos\theta(\hat{\mathbf{r}_1} + \hat{\mathbf{r}_2})]$$

but we know that

$$\hat{\mathbf{r}_1} + \hat{\mathbf{r}_2} = 2a = 2\sin\theta\hat{\mathbf{r}} \qquad r_1 = \sqrt{r^2 + h^2}$$

and since we can see from the diagram that

$$\cos\theta = \frac{h}{r_1} = \frac{h}{\sqrt{r^2+h^2}} \quad \sin\theta = \frac{r}{r_1} = \frac{r}{\sqrt{r^2+h^2}}$$

using this we now find that the total magnetic field is

$$\mathbf{B}_T = -\frac{3\mu_0 hrm}{2\pi(r^2+h^2)^{5/2}}\hat{\mathbf{r}}$$

To find out what the surface current is we use

$$\mathbf{B} = \mu_0(\mathbf{K} \times \hat{\mathbf{z}})$$

in order to solve for **K** we must take the cross products of both sides give respect to $\hat{\mathbf{z}}$ we find

$$\hat{\mathbf{z}} \times \mathbf{B} = \hat{\mathbf{z}} \times (\mathbf{K} \times \hat{\mathbf{z}})$$

using the following relationship

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

we find

$$\hat{\mathbf{z}} \times \mathbf{B} = \mu_0[\mathbf{K}(\hat{\mathbf{z}} \cdot \hat{\mathbf{z}}) - \hat{\mathbf{z}}(\hat{\mathbf{z}} \cdot \mathbf{K})$$

we know that

$$\hat{\mathbf{z}} \cdot \mathbf{K} = 0 \quad \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = 1$$

because they are orthogonal. We know that

$$\hat{\mathbf{z}} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{r}} & \hat{\phi} & \hat{\mathbf{z}} \\ 0 & 0 & 1 \\ -\frac{3\mu_0 hrm}{2\pi(r^2+h^2)^{5/2}} & 0 & 0 \end{vmatrix} = -\frac{3\mu_0 hrm}{2\pi(r^2+h^2)^{5/2}}\hat{\phi}$$

thus we find that the surface current is given by

$$\boxed{\mathbf{K} = -\frac{3hrm}{2\pi(r^2+h^2)^{5/2}}\hat{\phi}}$$

Problem # 9

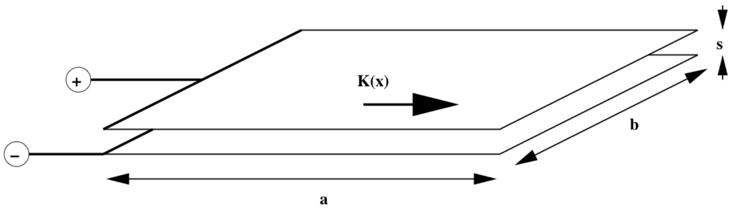


Figure 1: A parallel plate capacitor is connected on one side to a voltage source that increases the voltage across the capacitor linearly in time.

Figure 1 shows a parallel plate capacitor that is connected at one end ($x = 0$) to a source whose voltage increases slowly and linearly with time as $dV/dt = k$. Assume that edge effects are negligible in this problem, i.e. that $a, b \gg s$.

a) Find the surface current **K** in the top capacitor plate (positive voltage) as a function of position x along the capacitor.

Since we know that

$$Q(t) = CV$$

then

$$\frac{dQ(t)}{dt} = C\frac{dV}{dt} = Ck$$

thus

$$I = Ck$$

the capacitance from a parallel plate capacitor is given as

$$C = \frac{A\varepsilon_0}{s}$$

thus the current is given as

$$I = \frac{xb\varepsilon_0 k}{s}$$

In order to satisfy the boundary conditions we must write this as

$$I = \frac{(a-x)b\varepsilon_0 k}{s}$$

and we know that the surface current is defined as

$$\boxed{\mathbf{K} = \frac{I}{b}\hat{\mathbf{x}} = \frac{k\varepsilon_0(a-x)}{s}\hat{\mathbf{x}}}$$

b) Find the magnetic field **B** inside the capacitor by two different ways: using the integral Ampere/Faraday law for two different loops such that one has a flux only of real current and the other only of displacement current. From your expression for **B** (or by other means) determine the vector potential **A** inside the capacitor.

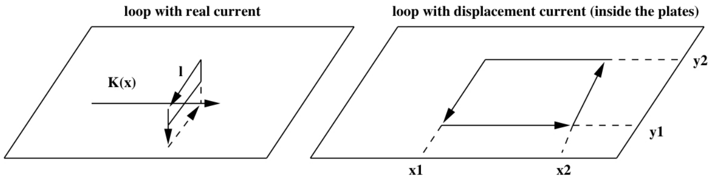
We know that Maxwell's equation is given by

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

and the integral Ampere/Faraday law can be written as

$$\nabla \times \mathbf{B} = \oint \mathbf{B} \cdot d\mathbf{l} = \int_A \mu_0 \mathbf{J} \cdot d\mathbf{a} + \mu_0 \varepsilon_0 \int_A \frac{\partial \mathbf{E}}{\partial t} \cdot d\mathbf{a}$$

the Amperian loops used are given in the following diagram



(a) (b)

For finding the magnetic field using the real current we can use the diagram on the left, we can see that since the loop is parallel with

$$\frac{\partial \mathbf{E}}{\partial t} = 0$$

Thus we have

$$\oint \mathbf{B} \cdot d\mathbf{l} = \int_A \mu_0 \mathbf{J} \cdot d\mathbf{a} = \mu_0 I_{enc} = \mu_0 K(x) l$$

thus we find that the magnetic field is given by

$$\mathbf{B} = \frac{\mu_0 \varepsilon_0 k}{s}(a-x)\hat{\mathbf{y}} \quad \mathbf{B} = 0 \text{ outside of the plate no z component)}$$

now to find this using the displacement current we know that

$$\mathbf{J} = 0 \quad \text{no surface current enclosed}$$

thus we find

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \varepsilon_0 \int_A \frac{\partial \mathbf{E}}{\partial t} \cdot d\mathbf{a} = -\frac{\mu_0 \varepsilon_0 k}{s}(x_1 - x_2)(y_1 - y_2)$$

where we can also write the left hand side as

$$B(x_1) - B(x_2) = \frac{\mu_0 \varepsilon_0 k}{s}(x_1 - x_2)\hat{\mathbf{y}}$$

where this can also be written as

$$\mathbf{B} = -\frac{\mu_0 \varepsilon_0 k}{s}(a-x)\hat{\mathbf{y}} \quad x_1 = x \;\; x_2 - a$$

which is the same thing we found from the real current, which it should be. To find what the vector potential is we can use

$$\mathbf{B} = \nabla \times \mathbf{A} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = -\left(\frac{\partial}{\partial x}A_z - \frac{\partial}{\partial z}A_x\right)\hat{\mathbf{y}} = \frac{\mu_0 \varepsilon_0 k}{s}(a-x)\hat{\mathbf{y}}$$

Thus we can see that a suitable solution of the vector potential could be given as

$$A_z = -\frac{\mu_0 \varepsilon_0 k}{s}(ax - \frac{1}{2}x^2)\hat{\mathbf{z}} \quad A_x = 0$$

Thus the magnetic field inside the plate and the vector potential can be written as

$$\boxed{\mathbf{B} = -\frac{\mu_0 \varepsilon_0 k}{s}(a-x)\hat{\mathbf{y}} \quad \mathbf{A} = -\frac{\mu_0 \varepsilon_0 k}{s}(ax - \frac{1}{2}x^2)\hat{\mathbf{z}}}$$

c) Find **B** and **A** outside the capacitor.

For part be we concluded that

$$\mathbf{B} = 0$$

for the vector potential outside we can guess a solution and see if it satisfies the boundary conditions we can use

$$\mathbf{A}_{\text{above}} = -\frac{\mu_0 \varepsilon_0 k}{s}(az - zx)\hat{\mathbf{x}} - \frac{\mu_0 \varepsilon_0 k}{s}(ax - \frac{1}{2}x^2 + \frac{1}{2}z^2)\hat{\mathbf{z}}$$

does this satisfy our boundary conditions?

$$\nabla \times \mathbf{A} = 0$$

we can see that this is satisfied,

$$\nabla \cdot \mathbf{A} = 0$$

this is satisfied if we assume that we are using the Coulomb gauge which is defined as this. What about

$$\mathbf{A}_{\text{above}}(z=0) = \mathbf{A}_{\text{below}}(z=0)$$

we can see that this is also satisfied. And finally

$$\frac{\partial \mathbf{A}_{\text{above}}(z=0)}{\partial n} - \frac{\partial \mathbf{A}_{\text{below}}(z=0)}{\partial n} = -\mu_0 K$$

and we can see that this will also be satisfied by our vector potential. Thus we can conclude that

$$\boxed{\mathbf{B}_{\text{above}} = 0 \quad \mathbf{A}_{\text{above}} = -\frac{\mu_0 \varepsilon_0 k}{s}(az - ax)\hat{\mathbf{x}} - -\frac{\mu_0 \varepsilon_0 k}{s}(ax - \frac{1}{2}x^2 + \frac{1}{2}z^2)\hat{\mathbf{z}}}$$

Problem # 10 Magnetic pressure on a solenoid

Consider a long solenoid that produces a magnetic field of magnitude **B** in its interior.

a) Considering the magnetic energy per unit length in a solenoid of radius R and $R + dR$, show that there is an outward pressure on the surface of the solenoid (i.e. on the many turns of wire that resemble a continuous sheet of current) of magnitude $P = B^2/(2\mu_0)$.

We know that the energy stored due to a magnetic field is

$$W = \frac{1}{2\mu_0}\int B^2 d\tau = \frac{B^2}{2\mu_0}V$$

where the volume of this solenoid is given by

$$dV = \pi h[(R+dR)^2 - R^2]$$

thus the energy per unit length is given by

$$\frac{dW}{h} = \frac{\pi B^2}{2\mu_0}[(R+dR)^2 - R^2]$$

We also know that the energy is related to the force by

$$dW = \int \mathbf{F}\cdot d\mathbf{l} = P\int A dr = 2\pi hP\int_{R}^{R+dR} r dr = \pi hP[(R+dR)^2 - R^2]$$

and we can see that this can also be written as

$$\frac{dW}{h} = P\pi[(R+dR)^2 - R^2] = \frac{B^2}{2\mu_0}\pi[(R+dR)^2 - R^2]$$

thus we find that the pressure is given by

$$\boxed{P = \frac{B^2}{2\mu_0}}$$

b) Suppose a public utility proposes to moderate the ebb and flow of intermittent power sources by storing energy in a large inductor, reaching a maximum field of, say, 14 Tesla. What is the storage capacity of this inductor, in kilowatt-hours / (meter)^3 ? What would be the magnetic pressure on this solenoid in atmospheres?

We know that

$$W = \frac{B^2}{2\mu_0}V \quad \frac{W}{V} = \frac{B^2}{2\mu_0}$$

and since we have been given the magnetic field and we know what μ_0 is we find

$$\frac{W}{V} = 7.8\times 10^7 \frac{\mathrm{J}}{\mathrm{m}^3} = \boxed{21.7\frac{\mathrm{kWh}}{\mathrm{m}^3} = 770 \text{ atm}}$$

c) Alternately, suppose the energy is stored instead in a large capacitor with, say, a dielectric constant $\varepsilon/\varepsilon_0 = 3$ and a maximum allowable field of 1.5×10^8 V/m. This storage would not require, for instance, cryogenic cooling of a superconductor to avoid resistive heating of a current-carrying solenoid. What is the storage capacity of this capacitor, also in kilowatt- hours / (meter)^3 ? For reference, the energy content of gasoline is about 100 kilowatt-hours / (meter)^3 .

We know that the energy stored in a capacitor with a dialectric media is given as

$$W = \frac{\varepsilon_0}{2}\int \varepsilon_r E^2 d\tau = \frac{\varepsilon_0\varepsilon_r}{2}E^2 V$$

thus

$$\frac{W}{V} = \frac{\varepsilon_0\varepsilon_r E^2}{2}$$

We know that

$$\frac{\varepsilon_0}{\varepsilon} = \varepsilon_r = 3$$

thus

$$\frac{W}{V} = \frac{3}{2}\varepsilon_0 E^2$$

given that we know what E is we find

$$\frac{W}{V} = 3\times 10^5 \frac{\mathrm{J}}{\mathrm{m}^3} = \boxed{.083\frac{\mathrm{kWh}}{\mathrm{m}^3}}$$

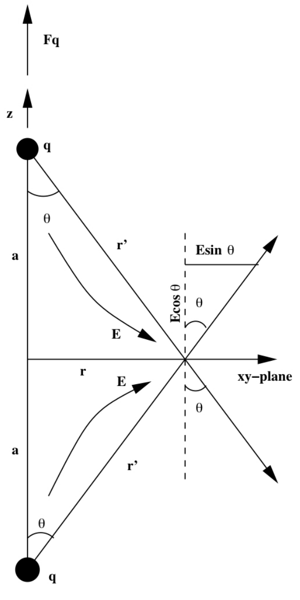
Chapter 8

Conservation Laws

8.1 Problems and Solutions

Problem # 1

a) Consider two equal point charges q, seperated by a distance $2a$. Construct the plane equidistant from the two charges. By integrating Maxwell's stress tensor over this plane, determine the force of one charge on the other.



Two positive charges

We can see that this configuration will cause a repulsive force on the top charge due to the bottom charge. If we pick a point located a distance r from the origin a distance r' from the charge and at $z = 0$ we see that the electric field here only has a $\hat{\mathbf{y}}$ component (the vertical components cancel by symmetry).

$$E_x = E_z = 0 \quad E_y(r') = 2E\sin\theta$$

Maxwell's stress tensor is given as

$$T_{ij} \equiv \varepsilon_0\left(E_iE_j - \frac{1}{2}\delta_{ij}E^2\right) + \frac{1}{\mu_0}\left(B_iB_j - \frac{1}{2}B^2\right) \tag{8.1}$$

The indices i and j refer to the coordinates x, y, and z, so the stress tensor has a total of nine components. We can write the stress tensor explicitily in matrix notation as

$$\vec{\mathbf{T}} = \begin{bmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{bmatrix}$$

One can form the dot product with a vectro **a**:

$$(\mathbf{a} \cdot \vec{\mathbf{T}})_j = \sum_{i=x,y,z} a_i T_{ij}$$

If the force per unit volume is given as

$$\mathbf{f} = \nabla \cdot \vec{\mathbf{T}} - \varepsilon_0 \mu_0 \frac{\partial \mathbf{S}}{\partial t}$$

then the total force is given by

$$\mathbf{F} = \oint \vec{\mathbf{T}} \cdot d\mathbf{a} - \varepsilon_0 \mu_0 \frac{d}{dt} \int_V \mathbf{S} d\tau \tag{8.2}$$

where we used the divergence theorem to convert the first term to a surface integral. In static situations the second term drops out and we are left with

$$\mathbf{F} = \oint \vec{\mathbf{T}} \cdot d\mathbf{a} \tag{8.3}$$

We can see from our figure that if we were to use the xy plane as our boundary surface then

$$d\mathbf{a} = r dr d\phi (\hat{\mathbf{z}})$$

We can write Maxwell's stress tensor as

$$T_{xx} = \frac{1}{2}\varepsilon_0 (E_x^2 - E_y^2 - E_z^2) \quad T_{yy} = \frac{1}{2}\varepsilon_0 (E_y^2 - E_z^2 - E_x^2) \quad T_{zz} = \frac{1}{2}\varepsilon_0 (E_z^2 - E_x^2 - E_y^2) \tag{8.4}$$

Since we know the net force is in the $\hat{\mathbf{z}}$ dircetion it suffices to calculate

$$(\vec{\mathbf{T}} \cdot d\mathbf{a})_z = T_{zx} da_x + T_{zy} da_y + T_{zz} da_z$$

since we know that

$$da_x = da_y = 0$$

this becomes

$$(\vec{\mathbf{T}} \cdot d\mathbf{a})_z = T_{zz} da_z \tag{8.5}$$

We can see that

$$T_{zz} = -\frac{\varepsilon_0}{2} E_y^2 \quad E_z = E_x = 0 \quad da_z = r dr d\phi$$

Thus the force is given Equation 3 and Equation 4 as

$$\mathbf{F} = -\frac{\varepsilon_0}{2} \oint E_y^2 r dr d\phi$$

we find that the electric field at r and r' is given by

$$E_y = 2E \sin\theta \quad \sin\theta = \frac{r}{r'} \quad \sin^2\theta = \frac{r^2}{r'^2}$$

The electric field for a uniform charge is given by

$$\mathbf{E}(\mathbf{r}') = \frac{q}{4\pi\varepsilon_0}\frac{1}{r'^2}\hat{\mathbf{r}}$$

since we only have an electric field in the $\hat{\mathbf{y}}$ direction this becomes

$$E_y = \frac{2q}{4\pi\varepsilon_0}\frac{1}{r'^2}\sin\theta\hat{\mathbf{y}}$$

If we integrate over the entire xy plane and also a hemisphere extending out to infinity, where $E = 0$ at infinity and thus the integral contributes nothing allows us to write the force as

$$\mathbf{F} = -\frac{\varepsilon_0}{2}\int_0^{2\pi} d\phi \int_0^{\infty}\left(\frac{2q}{4\pi\varepsilon_0}\frac{1}{r'^2}\sin\theta\right)^2 r dr$$

This simplifies to

$$\mathbf{F} = -\frac{q^2}{4\pi\varepsilon_0}\int_0^{\infty}\frac{r^3}{r'^6}dr$$

From the diagram we can see that $r' = \sqrt{a^2+r^2}$ thus

$$\mathbf{F} = -\frac{q^2}{4\pi\varepsilon_0}\int_0^{\infty}\frac{r^3}{(a^2+r^2)^3}dr = -\frac{q^2}{4\pi\varepsilon_0}\left[-\frac{2r^2+a^2}{4(r^2+a^2)^2}\right]_0^{\infty}$$

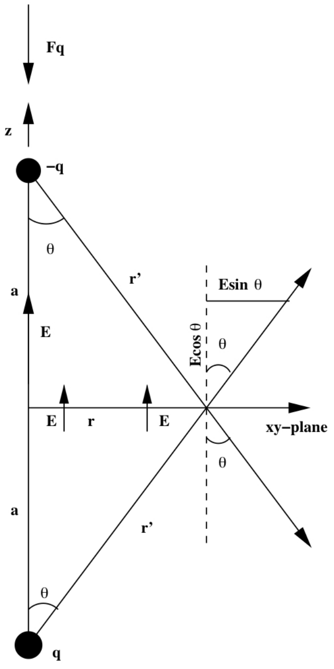
Evaluating this integral gives

$$\mathbf{F} = \boxed{\frac{q^2}{16\pi\varepsilon_0 a^2}\hat{\mathbf{z}}}$$

Which is exactly what we would expect using

$$\mathbf{F} = \frac{q^2}{4\pi\varepsilon_0 r^2}\hat{\mathbf{z}} \quad r = 2a \quad \mathbf{F} = \frac{q^2}{16\pi\varepsilon_0 a^2}\hat{\mathbf{z}}$$

b) Do the same for charges that are opposite in sign.



positive charge and negative charge

We can see that for this configuration we find the net force is attractive and if we consider the top charge to be $-q$ than it will experience a net force in the $-\hat{\mathbf{z}}$ direction. We can see that the electric field at r and r' is now given as

$$E_x = E_y = 0 \quad E_z = 2E\cos\theta$$

since we know that the net force is still in the $\hat{\mathbf{z}}$ direction then we can choose

$$d\mathbf{a} = rdrd\phi(\hat{\mathbf{z}})$$

and thus we it works to solve for

$$(\vec{\mathbf{T}} \cdot d\mathbf{a})_z = T_{zz}da_z$$

where

$$T_{zz} = \frac{\varepsilon_0}{2}E_z^2 \quad da_z = rdrd\phi$$

and so we find the force to be

$$\mathbf{F} = \frac{\varepsilon_0}{2}\int_0^{2\pi} d\phi \int_0^{\infty} \left(\frac{2q}{4\pi\varepsilon_0}\frac{1}{r'^2}\cos\theta\right)^2 rdr$$

where

$$\cos\theta = \frac{a}{r'} \quad \cos^2\theta = \frac{a^2}{r'^2}$$

this now gives

$$\mathbf{F} = \frac{q^2a^2}{4\pi\varepsilon_0}\int_0^{\infty}\frac{r}{(a^2+r^2)^3}dr$$

doing a u substitutin gives

$$u = a^2 + r^2 \;\; rdr = \frac{du}{2}$$

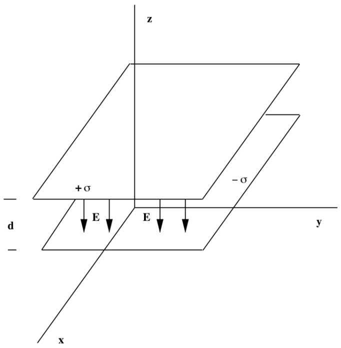
gives

$$\mathbf{F} = \frac{q^2a^2}{8\pi\varepsilon_0}\int_{a^2}^{\infty} u^{-3}du = \boxed{-\frac{q^2}{16\pi\varepsilon_0 a^2}\hat{\mathbf{z}}}$$

which is exactly what is expected.

Problem # 2

Consider an infinite parallel-plate capacitor, with the lower plate (at $z = -d/2$) carrying the charge density $-\sigma$, and the upper plate (at $z = +d/2$) carrying the charge density $+\sigma$.



a) Determine all nine elements of the stress tensor, in the region between the plates. Display your answer as a 3×3 matrix:

$$\begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix}$$

since the Maxwell stress equation is given by

$$T_{ij} \equiv \varepsilon_0 \left(E_i E_j - \frac{1}{2}\delta_{ij} E^2 \right) + \frac{1}{\mu_0} \left(B_i B_i - \frac{1}{2} B^2 \right)$$

and assuming that the magnetic field inside the capacitor is $\mathbf{B} = 0$ we can write the nine components of the stress tensor as

$$\vec{\mathbf{T}} = \varepsilon_0 \begin{pmatrix} E_x E_x - \frac{1}{2}\delta_{xx} E^2 & E_x E_y & E_x E_z \\ E_y E_x & E_y E_y - \frac{1}{2}\delta_{yy} E^2 & E_y E_z \\ E_z E_x & E_z E_y & E_z E_z - \frac{1}{2}\delta_{zz} E^2 \end{pmatrix}$$

this can be written as

$$\vec{\mathbf{T}} = \varepsilon_0 \begin{pmatrix} \frac{1}{2}\left(E_x^2 - E_y^2 - E_z^2\right) & E_x E_y & E_x E_z \\ E_y E_x & \frac{1}{2}\left(E_y^2 - E_z^2 - E_x^2\right) & E_y E_z \\ E_z E_x & E_z E_y & \frac{1}{2}\left(E_z^2 - E_x^2 - E_y^2\right) \end{pmatrix}$$

Looking at the figure above we can conclude that

$$E_x = E_y = 0 \quad E_z = -\frac{\sigma}{\varepsilon_0}\hat{\mathbf{z}}$$

The field in this configuration is pointing in the - $\hat{\mathbf{z}}$ direction this gives us

$$\boxed{\vec{\mathbf{T}} = \frac{\sigma^2}{2\varepsilon_0} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

b) Use Equation 8.22

$$\mathbf{F} = \oint_S \vec{\mathbf{T}} \cdot d\mathbf{a} - \varepsilon_0 \mu_0 \frac{d}{dt} \int_V \mathbf{S} d\tau$$

to determine the force per unit area on the top plate. Compare Eq 2.51

Given the static situation

$$\mathbf{S} = 0 \quad d\mathbf{a} = -\hat{\mathbf{z}}$$

we find that the force is given by

$$\mathbf{f} = \frac{\mathbf{F}}{d\mathbf{a}} = \frac{\sigma^2}{2\varepsilon_0} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \hat{\mathbf{n}}$$

since we know the force is only in the -$\hat{\mathbf{z}}$ direction we find

$$\boxed{\mathbf{f} = \vec{\mathbf{T}} = -\frac{\sigma^2}{2\varepsilon_0}\hat{\mathbf{z}}}$$

which is identical to Equation 2.51.

c) What is the momentum per unit area, per unit time, crossing the *xy* plane (or any other plane parallel to that one, between the plates).

According to Newton's second law. the force on an object is equal to the rate of change of its momentum

$$\mathbf{F} = \frac{d\mathbf{p}_{\text{mech}}}{dt}$$

where we can now write Equation 2 as

$$\frac{d\mathbf{p}_{\text{mech}}}{dt} = \oint_S \vec{\mathbf{T}} \cdot d\mathbf{a} - \varepsilon_0 \mu_0 \frac{d}{dt} \int_V \mathbf{S} d\tau \tag{8.6}$$

The second integral represents $momentum\,stored\,in\,the\,electromagnetic\,fields\,themselves$:

$$\mathbf{p}_{\text{mech}} = \mu_0 \varepsilon_0 \int_V \mathbf{S} d\tau$$

while the first integral is the $momentum\ per\ unit\ time\ flowing\,through\,the\,surface$. Equation 6 is the general statement of $conservation\,of\ momentum$ in electrodynamics. If we let $\mathcal{P}_{\text{mech}}$ be the density of *mechanical* momentum, and $\mathcal{P}_{\text{em}}$ the density of momentum in the fields

$$\mathcal{P}_{\text{em}} = \mu_0 \varepsilon_0 \mathbf{S}$$

then equation 6 in differential form says

$$\frac{\partial}{\partial t}(\mathcal{P}_{\text{mech}} + \mathcal{P}_{em}) = \nabla \cdot \vec{\mathbf{T}}$$

and so we call -$\vec{\mathbf{T}}$ the **momentum flux density**. We have found this to be

$$\mathcal{P}_{\text{flux density}} = -\vec{\mathbf{T}} = -\mathbf{f} = \boxed{\frac{\sigma^2}{2\varepsilon_0}\hat{\mathbf{z}}}$$

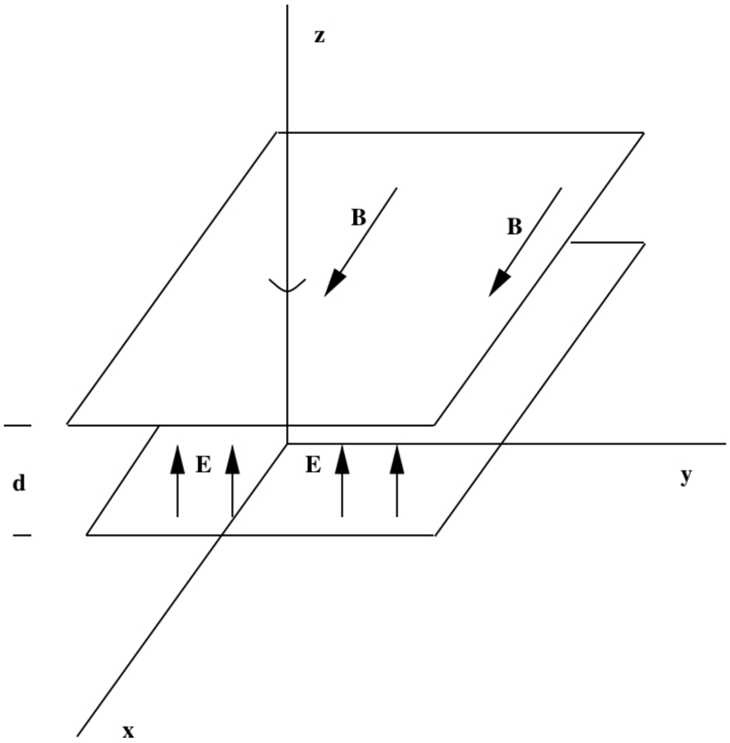
d). At the plates this momentum is aborbed, and the plate recoils (unless there is some non-electrical force holding them in position). Find the recoil force per unit area on the top plate, and compare your answer to (b). [$Note$: This is not an $additional$ force, but rather an alternative way of calculating the *same* force- in (b) we got it from the force law, and in (d) we get it by conservation of momentum.]

The recoil force on the top plate should be equal and opposite to the force exerted on the plate

$$\frac{\mathbf{F}_{\text{recoil}}}{\mathbf{a}} = -\frac{d\mathbf{p}_{\text{mech}}}{dadt} = -\mathcal{P}_{\text{flux density}} = -\frac{\sigma^2}{2\varepsilon_0}\hat{\mathbf{z}}$$

which is the same as (b), like Griffith's implied in the problem. Same problem just two different ways of looking at it.

Problem # 3



A charged parallel-plate capacitor (with uniform electric field $\mathbf{E} = E\hat{\mathbf{z}}$) is placed in a uniform magnetic field $\mathbf{B} = B\hat{\mathbf{x}}$ as shown by the figure

a) Find the electromagnetic momentum in the space between the plates.

The electromagnetic momentum is defined as

$$\mathbf{p}_{\text{em}} = \mu_0\varepsilon_0 \int_V \mathbf{S}d\tau = \varepsilon_0 \int (\mathbf{E}\times\mathbf{B})d\tau$$

thus we find

$$\mathbf{p}_{\text{em}} = \varepsilon_0 EB\sin\theta V\hat{\mathbf{y}}$$

and since $\theta = \pi/2$ we find that the momentum is given as

$$\boxed{\mathbf{p}_{\text{em}} = \varepsilon_0 EBdA\hat{\mathbf{y}}}$$

where the volume is given by

$$V = dA$$

b) Now a resistive wire is connected between the plates, along the z axis, so that the capacitor slowely discharges. The current through the wire will experience a magnetic force; what is the total impulse delivered to the system, during the discharge?

We know that the impulse is given as

$$\mathbf{I} = \int \mathbf{F}dt$$

The magnetic force on a segment of current-carrying wire is given by

$$\mathbf{F}_{\text{mag}} = \int I(d\mathbf{l}\times\mathbf{B})$$

thus the impulse can now be written as

$$\mathbf{I} = \int_0^\infty \int_0^l I(d\mathbf{l} \times \mathbf{B})dt = \int_0^\infty I(\mathbf{l} \times \mathbf{B})dt$$

since we know that $\mathbf{l} = d\hat{\mathbf{z}}$ and $\mathbf{B} = B\hat{\mathbf{x}}$ gives us

$$\mathbf{I} = \int_0^\infty IdB(\hat{\mathbf{z}} \times \hat{\mathbf{x}})dt = |-dB\hat{\mathbf{y}}It|_0^\infty = -dB(q(\infty) - q(0)\hat{\mathbf{y}} = q(0)dB\hat{\mathbf{y}}$$

where $q(t) = It$ and $q(\infty) = 0$, we assume that there is no charge left at $t = \infty$. We also know that Gauss's law gives

$$\int \mathbf{E} \cdot d\mathbf{a} = \frac{q_{enc}}{\varepsilon_0} \quad EA\varepsilon_0 = q_{enc}$$

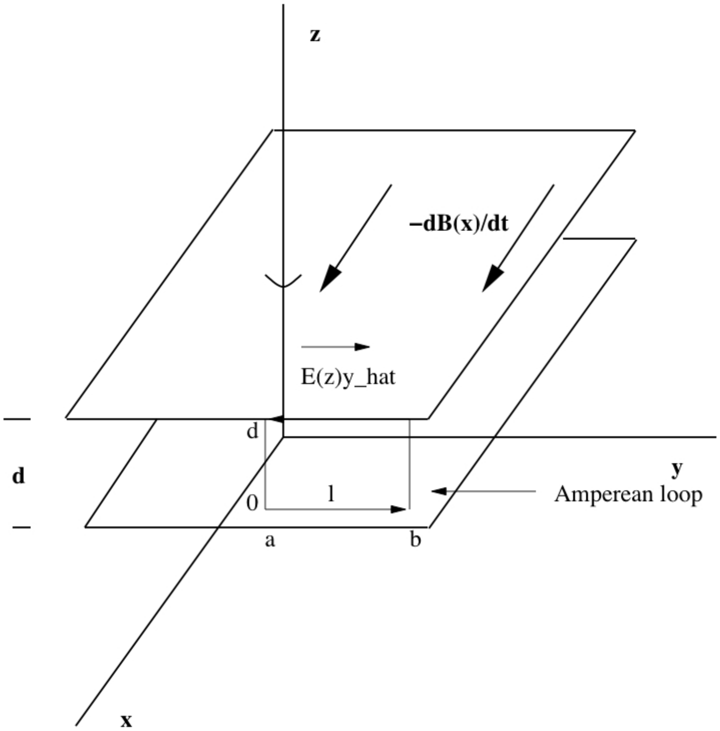
Thus we find that the impulse is given by

$$\boxed{\mathbf{I} = \varepsilon_0 EBdA\hat{\mathbf{y}}}$$

which is the same thing as

$$\mathbf{I} = \int \mathbf{F}dt = \int \left(\frac{d\mathbf{p}\text{em}}{dt}\right) dt = \mathbf{p}\text{em}$$

c) Instead of turning off the *electric* field (as in (b)), suppose we slowely reduce the *magnetic* field. This will induce a Faraday electric field, which in turn exerts a force on the plates. Show that the total impulse is (again) equal to the momentum originally stored in the fields.



Since we know that Faraday's law in integral form is given as

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{a}$$

and in differential form

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

we know that a changing magnetic field in the $\hat{\mathbf{x}}$ direction induces a Faraday electric field in the $\hat{\mathbf{y}}$ direction. Using the Amperean loop shown in the figure we find

$$\oint \mathbf{E} \cdot d\mathbf{l} = \int_a^b \mathbf{E} \cdot d\mathbf{l} + \int_0^d \mathbf{E} \cdot d\mathbf{l} - \int_a^b \mathbf{E} \cdot d\mathbf{l} - \int_0^d \mathbf{E} \cdot d\mathbf{l}$$

we can see that the second and last term cancel and since the electric field is in the $\hat{\mathbf{y}}$ direction we find

$$(E(0) + E(d))l = -\frac{\partial}{\partial t}(B(\infty) - B(0))dl = \frac{\partial B(0)dl}{\partial t}\hat{\mathbf{y}}$$

and the force is simply

$$\mathbf{F} = qE = q(E(0) + E(d)) = qd\left(\frac{\partial B(0)}{\partial t}\right)\hat{\mathbf{y}}$$

using this in the impulse equation yields

$$\mathbf{I} = \int \mathbf{F} dt = qd \int \left(\frac{\partial B(0)}{\partial t}\right) dt\hat{\mathbf{y}} = qdB\hat{\mathbf{y}}$$

but we have shown that

$$q = \varepsilon_0 AE$$

thus the impulse is

$$\boxed{\mathbf{I} = \varepsilon_0 EBdA\hat{\mathbf{y}}}$$

which is the same thing as in the momentum stored in the fields.

Problem # 4

Imagine an iron sphere of radius R that carries a charge Q and a uniform magnetization $\mathbf{M} = M\hat{\mathbf{z}}$. The sphere is initially at rest.

a) Compute the angular momentum stored in the electromagnetic fields.

Since we know that the magnetic field outside of a uniformly magnetized sphere is given by the magnetic dipole equation

$$\mathbf{B}_{\text{dip}} = \frac{\mu_0 m}{4\pi r^3}[2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\theta}]$$

and we know that the electric field inside this conducter is zero so we are only concerned with the electric field outside of this sphere which can be modeled as the electric field of a point charge at a large distance away

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}}$$

the angular momentum density is given as

$$\vec{\ell}_{\text{em}} = \mathbf{r} \times \mathcal{P}_{\text{em}} = \varepsilon_0[\mathbf{r} \times (\mathbf{E} \times \mathbf{B})]$$

where we have defined the momentum density

$$\mathcal{P}_{\text{em}} = \mu_0\varepsilon_0\mathbf{S} = \varepsilon_0(\mathbf{E} \times \mathbf{B})$$

thus we find

$$\mathbf{E}\times\mathbf{B} = \frac{\mu_0 qm}{16\pi^2\varepsilon_0 r^5}\sin\theta\hat{\phi}$$

the total angular momentum stored in the fields is given by

$$\mathbf{L}_{\text{em}} = \vec{\ell}_{\text{em}} V$$

where

$$\vec{\ell}_{\text{em}} = -\frac{\mu_0 qm}{16\pi^2 r^4}\sin\theta\hat{\theta}$$

we know that $\hat{\theta}$ in cartesian coordinates is

$$\hat{\theta} = \cos\theta\cos\phi\hat{\mathbf{x}} + \cos\theta\sin\phi\hat{\mathbf{y}} - \sin\theta\hat{\mathbf{z}}$$

thus the angular momentum density is

$$\vec{\ell}_{\text{em}} = -\frac{\mu_0 qm}{16\pi^2 r^4}\sin\theta(\cos\theta\cos\phi\hat{\mathbf{x}} + \cos\theta\sin\phi\hat{\mathbf{y}} - \sin\theta\hat{\mathbf{z}})$$

and the total angular momentum is

$$\mathbf{L}_{\text{em}} = \int \vec{\ell}_{\text{em}} d\tau = \frac{\mu_0 qm}{16\pi^2}\int_R^\infty \frac{1}{r^2}dr\int_0^{2\pi}d\phi\int_0^\pi \sin^3\theta d\theta\hat{\mathbf{z}}$$

which becomes

$$\mathbf{L}_{\text{em}} = \frac{4(2\pi)\mu_0 qm}{(3)16\pi^2 R}$$

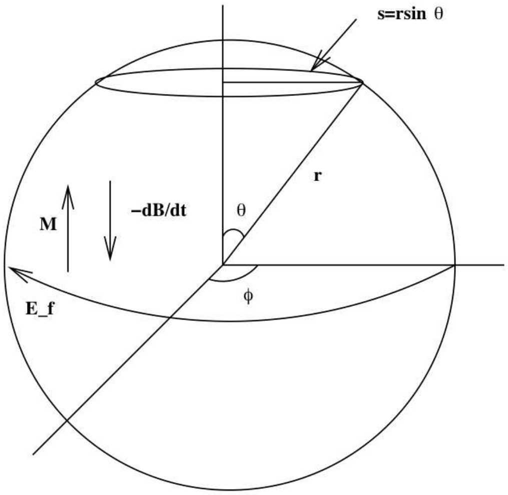
and if

$$m = \frac{4}{3}\pi R^3 M$$

gives

$$\boxed{\mathbf{L}_{\text{em}} = \frac{2}{9}\mu_0 MqR^2\hat{\mathbf{z}}}$$

b) Suppose the sphere is gradually (and uniformly) demagnetized (perhaps by heating it up past the Curie point). Use Faraday's law to determine the induced electric field, find the torque this field exerts, and calculate the total angualr momentum imparted to the sphere in the course of the demagnetization.



If we let

$$d\mathbf{l} = -2\pi s\hat{\phi} \quad d\mathbf{a} = \pi s^2\hat{\mathbf{z}}$$

thus we can use Faraday's law

$$\oint \mathbf{E}\cdot d\mathbf{l} = \int -\frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{a}$$

thus the Faraday electric field is given as

$$\mathbf{E}_{\text{far}} = -\frac{d\mathbf{B}}{dt}\frac{s^2}{2}\hat{\phi} = -\frac{1}{3}\mu_0\frac{dM}{dt}R^2\sin\theta\hat{\phi}$$

but we know that

$$s = R\sin\theta \quad \frac{d\mathbf{B}}{dt} = \frac{2}{3}\mu_0\frac{dM}{dt}$$

the torque is given as

$$\mathbf{N} = \mathbf{r}\times\mathbf{F}$$

where the force is from the induced Faraday electric field

$$\mathbf{F} = q\mathbf{E}_{\text{far}} \quad d\mathbf{F} = dq\mathbf{E}_{\text{far}} = \sigma d\mathbf{a}\mathbf{E}_{\text{far}}$$

and the total torque is given by

$$\int_0^N d\mathbf{N} = \int \mathbf{r}\times d\mathbf{F} = \int RdF(\hat{\mathbf{r}}\times\hat{\phi}) = \int -RdF(\hat{\theta}) = \int RdF\sin\theta\hat{\mathbf{z}}$$

if we now define

$$d\mathbf{a} = R^2\sin\theta d\theta d\phi$$

thus

$$\mathbf{N} = \frac{1}{3}\frac{\mu_0}{3}R^4\frac{dM}{dt}\sigma\int d\phi\int_0^\pi \sin^3\theta d\theta\hat{\mathbf{z}} = \frac{1}{3}\frac{2\sigma 4\pi R^2}{3}\frac{dM}{dt}R^2\mu_0\hat{\mathbf{z}}$$

thus the total torque is given by

$$\mathbf{N} = \frac{2}{9}\mu_0 qR^2\frac{dM}{dt}\hat{\mathbf{z}}$$

and thus the total angular momentum is given as

$$\mathbf{L}_{\text{mech}} = \mathbf{N}dt = \boxed{\frac{2}{9}q\mu_0R^2M\hat{\mathbf{z}}}$$

c) Suppose instead of *demagnetizing* the sphere we *discharge* it, by connecting a grounding wire to the north pole. assume the current flows over the surface in such a way that the charge density remains uniform. Use the Lorentz force law to determine the torque on the sphere, and calculate the total angular momentum imparted to the sphere in the course of the discharge. (The magnetic field is discontinuous at the surface... does this matter?)

We know that the force will now be given by

$$\mathbf{F} = q(\mathbf{v}\times\mathbf{B}) = (\mathbf{K}\times\mathbf{B})d\mathbf{a}$$

and we know that the torque is now given by

$$\mathbf{N} = \mathbf{r}\times\mathbf{F} = (\mathbf{r}\times(\mathbf{K}\times\mathbf{B}))d\mathbf{a}$$

using the BAC-CAB rule for the cross product of a cross product we find

$$\mathbf{N} = (\mathbf{K}(\mathbf{r}\cdot\mathbf{B}) - \mathbf{B}(\mathbf{r}\cdot\mathbf{K}))d\mathbf{a}$$

we know that $\mathbf{r}\cdot\mathbf{K} = 0$ due to orthogonality, thus we find

$$\mathbf{N} = \mathbf{K}(\mathbf{r}\cdot\mathbf{B})d\mathbf{a}$$

we know that the magnetic field will just be the average of the internal field and the external field

$$\mathbf{B} = \frac{1}{2}\left[\frac{\mu_0 m}{2\pi r^3}(\cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\theta}) + \frac{\mu_0 m\cos\theta R^3}{2\pi r^3}\hat{\mathbf{r}} + \frac{\mu_0 m\sin\theta R^3}{4\pi r^3}\hat{\theta}\right]$$

and

$$\mathbf{r}\cdot\mathbf{B} = \frac{\mu_0 m}{2\pi r^3}\cos\theta$$

now we need to find what the surface current is, we know that

$$q = \int\sigma d\mathbf{a} = \sigma\int_0^{2\pi}d\phi\int_\theta^\pi R^2\sin\theta d\theta = \sigma 2\pi R^2(1+\cos\theta)$$

and

$$\frac{dq}{dt} = \frac{d}{dt}[2\pi R^2\sigma(1+\cos\theta)]$$

we also know that

$$\sigma = \frac{Q}{4\pi R^2}$$

thus we find the current to be given by

$$I = \frac{dQ}{dt}\left(\frac{1+\cos\theta}{2}\right)$$

but we know that the surface current is given by

$$\mathbf{K} = \frac{I}{l\perp}\hat{\theta}$$

but

$$l_\perp = 2\pi R\sin\theta \quad I = -\frac{dQ}{dt}\left(\frac{1+\cos\theta}{2}\right)$$

thus the surface current is

$$\mathbf{K} = -\frac{dQ}{dt}\left(\frac{1+\cos\theta}{4\pi R\sin\theta}\right)\hat{\theta}$$

thus our total torque is given by

$$\mathbf{N} = -\frac{dQ}{dt}\int\left(\frac{1+\cos\theta}{4\pi R\sin\theta}\right)\left(\frac{\mu_0 m}{2\pi}\cos\theta\right)d\mathbf{a}\hat{\theta}$$

but we know that

$$d\mathbf{a} = R^2\sin\theta d\theta d\phi \qquad \hat{\theta} \to (-\sin\theta\hat{\mathbf{z}})$$

thus this integral be

$$\mathbf{N} = \frac{m\mu_0}{8R\pi^2}\frac{dQ}{dt}\int_0^{2\pi}d\phi\int_0^\pi(\sin\theta\cos\theta + \sin\theta\cos^2\theta)d\theta\hat{\mathbf{z}}$$

since we know that

$$\frac{m}{4\pi R^3} = \frac{M}{3}$$

thus we find

$$\mathbf{N} = \frac{\mu_0 M}{6\pi} R^2 \frac{dQ}{dt} \int_0^{2\pi} d\phi \int_0^{\pi} (\sin\theta\cos\theta + \sin\theta\cos^2\theta) d\theta \hat{\mathbf{z}}$$

which gives

$$\mathbf{N} = \frac{1}{3}\mu_0 R^2 M \frac{dQ}{dt} \int_0^{\pi} (\sin\theta\cos\theta + \sin\theta\cos^2\theta) d\theta$$

which becomes

$$\mathbf{N} = \frac{2}{9}\mu_0 M R^2 \frac{dQ}{dt}$$

and thus the total angular momentum is

$$\mathbf{L}_{\text{mech}} = \mathbf{N} dt = \boxed{\frac{2}{9}\mu_0 M R^2 Q \hat{\mathbf{z}}}$$

Chapter 9

Electromagnetic Waves

9.1 Problems and Solutions

Problem # 1

Suppose

$$\mathbf{E}(r,\theta,\phi,t) = A\frac{\sin\theta}{r}[\cos(kr-\omega t) - (1/kr)\sin(kr-\omega t)]\hat{\phi}, \quad \text{with } \frac{\omega}{k} = c$$

(This is, incidentally, the simplest possible **spherical wave**. For notational convenience, let $(kr-\omega t) \equiv u$ in your calculations.)

a) Show that **E** obeys all four of Maxwell's equations, in vacuum, and find the associated magnetic fields.

In regions of space where there is no charge or current, Maxwell's equations read

$$\begin{array}{lllll} \text{(i)} & \nabla\cdot\mathbf{E} & = & 0 & \text{(iii)} \;\; \nabla\times\mathbf{E} = -\dfrac{\partial\mathbf{B}}{\partial t} \\ \text{(ii)} & \nabla\cdot\mathbf{B} & = & 0 & \text{(iv)} \;\; \nabla\times\mathbf{B} = \mu_0\varepsilon_0\dfrac{\partial\mathbf{E}}{\partial t} \end{array}$$

for (i) we find

$$\nabla\cdot\mathbf{E} = \nabla\cdot A\frac{\sin\theta}{r}[\cos u - (1/kr)\sin u]\hat{\phi}$$

in spherical coordinates this gives

$$\nabla\cdot\mathbf{E} = \frac{A}{r^2}\frac{\partial}{\partial\phi}[\cos u - (1/kr)\sin u] = \boxed{0}$$

to find the magnetic field we must use

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$$

where letting

$$v_\phi = A\frac{\sin\theta}{r}[\cos u - (1/kr)\sin u]$$

$$\nabla\times\mathbf{E} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta v_\phi)\right]\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial}{\partial r}(rv_\phi)\hat{\theta}$$

which is given by

$$\begin{aligned}\nabla\times\mathbf{E} &= \frac{A2\sin\theta\cos\theta}{r^2\sin\theta}[\cos u-\frac{1}{kr}\sin u]\hat{\mathbf{r}}-\frac{A\sin\theta}{r}\frac{d}{dr}[\cos u-\frac{1}{kr}\sin u]\hat{\theta}\\ \nabla\times\mathbf{E} &= \frac{A2\cos\theta}{r^2}[\cos u-\frac{1}{kr}\sin u]\hat{\mathbf{r}}-\frac{A\sin\theta}{r}\left[-k\sin u+\frac{1}{kr^2}\sin u-k\cos u\right]\hat{\theta}\end{aligned}$$

we know that this is equal to

$$\frac{A2\cos\theta}{r^2}[\cos u-\frac{1}{kr}\sin u]\hat{\mathbf{r}}-\frac{A\sin\theta}{r}\left[-k\sin u+\frac{1}{kr^2}\sin u-k\cos u\right]\hat{\theta}=-\frac{\partial\mathbf{B}}{\partial t}$$

thus the magnetic field is given by

$$\mathbf{B}=\int\frac{A2\cos\theta}{r^2}[\cos u-\frac{1}{kr}\sin u]\hat{\mathbf{r}}dt-\int\frac{A\sin\theta}{r}\left[-k\sin u+\frac{1}{kr^2}\sin u-\frac{1}{r}\cos u\right]\hat{\theta}dt$$

solving this integral gives

$$\mathbf{B}=\boxed{\frac{2A\cos\theta}{\omega r^2}\left[\sin u+\frac{1}{kr}\cos u\right]\hat{\mathbf{r}}+\frac{A\sin\theta}{\omega r}\left[-k\cos u+\frac{1}{kr^2}\cos u+\frac{1}{r}\sin u\right]\hat{\theta}}$$

now for the divergence of the magnetic field

$$\begin{aligned}\nabla\cdot\mathbf{B} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{2A\cos\theta}{\omega r^2}\left[\sin u+\frac{1}{kr}\cos u\right]\right)\\ &\quad+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{A\sin^2\theta}{\omega r}\left[-k\cos u+\frac{1}{kr^2}\cos u+\frac{1}{r}\sin u\right]\right)\end{aligned}$$

which is simply 0

$$\boxed{\nabla\cdot\mathbf{B}=0}$$

now to for the last of Maxwell's equation we use

$$\nabla\times\mathbf{B}=\mu_0\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}$$

if we call

$$v_r=\frac{2A\cos\theta}{\omega r^2}\left[\sin u+\frac{1}{kr}\cos u\right]\quad v_\theta=\frac{A\sin\theta}{\omega r}\left[-k\cos u+\frac{1}{kr^2}\cos u+\frac{1}{r}\sin u\right]$$

Thus the cross product gives

$$\nabla\times\mathbf{B}=\frac{1}{r}\left[\frac{\partial}{\partial r}(rv_\theta)-\frac{\partial v_r}{\partial\theta}\right]\hat{\phi}$$

$$\frac{\partial}{\partial r}(rv_\theta)=\frac{A\sin\theta}{\omega}\left[k^2\sin u-\frac{1}{r^2}\sin u-\frac{2}{kr^3}\cos u-\frac{1}{r^2}\sin u+\frac{k}{r}\cos u\right]$$

and

$$\frac{\partial v_r}{\partial\theta}=\frac{2A}{\omega r^2}\left[\sin u+\frac{1}{kr}\cos u\right](-\sin\theta)k$$

thus summing up these two contributions yields

$$\nabla\times\mathbf{B}=\frac{A\sin\theta}{cr}\left[k\sin u+\frac{1}{r}\cos u\right]\hat{\phi}=\mu_0\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}$$

now we know that

$$\mu_0\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}=\frac{1}{c^2}\frac{\partial}{\partial t}\left[A\frac{\sin\theta}{r}[\cos u-(1/kr)\sin u\right]\hat{\phi}=\frac{1}{c^2}\frac{A\sin\theta}{r}\left[\frac{\omega}{k}k\sin u+\frac{\omega}{k}\frac{1}{r}\cos u\right]=\frac{A\sin\theta}{cr}\left[k\sin u+\frac{1}{r}\cos u\right]$$

thus we can see that both sides are equal.

b) Calculate the Poynting vector. Average **S** over a full cycle to get the intensity vector **I**. (Does it point in the expected direction? does it fall like r^{-2}, as it should?)

We know that the Poynting vector is given by

$$\mathbf{S}=\frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B})$$

given the magnetic field found in part (a) and using the electric field given gives (minus a lot of algebra)

$$\mathbf{S}=\frac{1}{\mu_0}\left[\begin{matrix}\hat{\mathbf{r}} & \hat{\theta} & \hat{\phi}\\ 0 & 0 & A\frac{\sin\theta}{r}[\cos(kr-\omega t)-(1/kr)\sin(kr-\omega t)]\\ \frac{2A\cos\theta}{\omega r^2}\left[\cos u+\frac{1}{kr}\sin u\right] & \frac{A\sin\theta}{\omega r}\left[-k\cos u+\frac{1}{kr^2}\cos u+\frac{1}{r}\sin u\right]\hat{\theta} & 0\end{matrix}\right.$$

which will give

$$\begin{aligned}\mathbf{S} &= -\frac{A^2\sin^2\theta}{\omega r^2}[\cos u-(1/kr)\sin u]\left[-k\cos u+\frac{1}{kr^2}\cos u+\frac{1}{r}\sin u\right]\hat{\mathbf{r}}\\ &\quad+\frac{2A^2\sin\theta\cos\theta}{\omega r^3}[\cos u-(1/kr)\sin u]\left[\cos u+\frac{1}{kr}\sin u\right]\hat{\theta}\end{aligned}$$

Simplifying the $\hat{\mathbf{r}}$ component gives

$$\mathbf{S}(\hat{\mathbf{r}})=-\frac{A^2\sin\theta^2}{\mu_0\omega r^2}\left[\left(\frac{2}{r}-\frac{1}{k^2r^3}\right)\frac{\sin 2u}{2}+\frac{1}{kr^2}\cos 2u-k\cos^2 u\right]\hat{\mathbf{r}}$$

we know that the intensity is given by

$$\mathbf{I}=\langle\mathbf{S}\rangle$$

but we also know that

$$\langle\sin u\cos u\rangle=\langle\sin 2u\rangle=\langle\cos 2u\rangle=0\quad\langle\cos^2 u\rangle=\langle\sin^2 u\rangle=\frac{1}{2}$$

we find that the intensity will now be given by

$$\mathbf{I}=\langle\mathbf{S}\rangle=\boxed{\frac{A^2k\sin\theta^2}{2\mu_0\omega r^2}\hat{\mathbf{r}}}$$

c) Integrate $\mathbf{I}\cdot d\mathbf{a}$ over s spherical surface to determine the total power radiated **[***Answer* : $4\pi A^2/3\mu_0 c$**]**

we know that

$$P = \oint \mathbf{I} \cdot d\mathbf{a} = \frac{A^2 k}{2\mu_0 \omega} \int_0^{2\pi} d\phi \int_0^{\pi} \sin^3\theta = \frac{4}{3} \frac{A^2 2\pi k}{2\mu_0 \omega} = \boxed{\frac{4\pi A^2}{3\mu_0 c}}$$

Problem # 2

Find the fields, and charge and current distributions, corrsponding to

$$V(\mathbf{r},t) = 0, \quad \mathbf{A}(\mathbf{r},t) = -\frac{1}{4\pi\varepsilon_0} \frac{qt}{r^2} \hat{\mathbf{r}}$$

We know that

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} = \boxed{\frac{q}{4\pi\varepsilon_0 r^2} \hat{\mathbf{r}}}$$

and the magnetic field is given by

$$\mathbf{B} = \nabla \times \mathbf{A} = \boxed{0}$$

to find the charge distribution we can take the divergence of the electric field

$$\rho = \varepsilon_0 \nabla \cdot \mathbf{E} = \frac{q}{4\pi} \nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2} \right) = \boxed{q\delta^3(\mathbf{r})}$$

and finally the current density

$$\mathbf{J} = \frac{1}{\mu_0} \nabla \times \mathbf{B} - \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \boxed{0}$$

since we know that the magnetic field is 0 and also that the electric field is constant in time.

Problem # 3

Use the gauge function $\lambda = -(1/4\pi\varepsilon_0)(qt/r)$ to transform the potentials in Prob. 10.3 and comment on the result.

Using Equation 10.7 we find

$$\mathbf{A}' = \mathbf{A} + \nabla\lambda = 0 \quad V' = V - \frac{\partial \lambda}{\partial t}$$

thus

$$\mathbf{B} = \nabla \times \mathbf{A}' = \boxed{0}$$

we can also see that

$$\mathbf{E} = -\nabla V' - \frac{\partial \mathbf{A}'}{\partial t} = \boxed{\frac{q}{4\pi\varepsilon_0 r^2} \hat{\mathbf{r}}}$$

and the current distribution is again

$$\mathbf{J} = \boxed{0}$$

These are the same conclusions, this follows from the fact that we can add any $\nabla\lambda$ to $\mathbf{A}$, provided we simultaneously subtract $\partial\lambda/\partial t$ from V.

Problem # 4

Find all the elements of the Maxwell stress tensor for a monochromatic plane wave traveling in the z direction and linearly polarized in the x direction

$$\tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 e^{i(kx-\omega t)} \hat{\mathbf{x}} \quad \tilde{\mathbf{B}}(z,t) = \frac{1}{c} \tilde{\mathbf{E}}_0 e^{i(kx-\omega t)} \hat{\mathbf{y}}$$

or taking the real part

$$\mathbf{E}(z,t) = E_0 \cos(kz - \omega t + \delta)\hat{\mathbf{x}} \quad \mathbf{B}(z,t) = \frac{1}{c} E_0 \cos(kz - \omega t + \delta)\hat{\mathbf{y}}$$

Does your answer make sense? (Remember that $\overline{\mathbf{T}}$ represents the momentum flux densitu.) How is the momentum flux density related to the energy density in this case?

$$T_{ij} \equiv \varepsilon_0 \left(E_i E_j - \frac{1}{2}\delta_{ij} E^2 \right) + \frac{1}{\mu_0} \left(B_i B_j - \frac{1}{2}\delta_{ij} B^2 \right) \tag{9.1}$$

The indices i and j refer to the coordinates x, y, and z, so the stress tensor has a total of nine components. We can write the stress tensor explicitily in matrix notation as

$$\vec{\mathbf{T}} = \begin{bmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{bmatrix}$$

We can write Maxwell's stress tensor as

$$\begin{aligned} T_{xx} &= \frac{1}{2}\varepsilon_0 (E_x^2 - E_y^2 - E_z^2) + \frac{1}{2\mu_0}(B_x^2 - B_y^2 - B_z^2) \qquad (9.2) \\ T_{yy} &= \frac{1}{2}\varepsilon_0 (E_y^2 - E_z^2 - E_x^2) + \frac{1}{2\mu_0}(B_y^2 - B_z^2 - B_x^2) \qquad (9.3) \\ T_{zz} &= \frac{1}{2}\varepsilon_0 (E_z^2 - E_x^2 - E_y^2) + \frac{1}{2\mu_0}(B_z^2 - B_x^2 - B_y^2) \qquad (9.4) \end{aligned}$$

and

$$T_{xy} = T_{xz} = T_{yx} = T_{yz} = T_{zx} = T_{zy} = 0$$

and since we know

$$B_x = B_z = E_y = E_z = 0$$

$$\begin{aligned} T_{xx} &= \frac{1}{2}\varepsilon_0 E_x^2 - \frac{1}{2\mu_0} B_y^2 \qquad (9.5) \\ T_{yy} &= -\frac{1}{2}\varepsilon_0 E_x^2 + \frac{1}{2\mu_0} B_y^2 \qquad (9.6) \\ T_{zz} &= -\frac{1}{2}\varepsilon_0 E_x^2 - \frac{1}{2\mu_0} B_y^2 \qquad (9.7) \end{aligned}$$

thus Maxwell's stress tensor in matrix notaion is given by

$$\vec{\mathbf{T}} = \begin{bmatrix} \frac{1}{2}\varepsilon_0 E_x^2 - \frac{1}{2\mu_0} B_y^2 & 0 & 0 \\ 0 & -\frac{1}{2}\varepsilon_0 E_x^2 + \frac{1}{2\mu_0} B_y^2 & 0 \\ 0 & 0 & -\frac{1}{2}\varepsilon_0 E_x^2 - \frac{1}{2\mu_0} B_y^2 \end{bmatrix}$$

where the off diagonal elements are all zero. Plugging in our expressions for the electric field and magnetic field gives

$$\vec{\mathbf{T}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\varepsilon_0 E_0^2 \cos^2(kz - \omega t + \delta) \end{bmatrix}$$

this makes sense because the wavefront is moving in the z direction. In this case the momentum density is equal to the negative of the momentum density. Griffiths defines the energy density on Equation 9.55 as

$$u = \varepsilon_0 E_0^2 \cos^2(kz - \omega t + \delta)$$

which is

$$u = -\bar{\mathbf{T}} = -T_{zz} = \varepsilon_0 E_0^2 \cos^2(kz - \omega t + \delta)$$

Problem # 5

a) Suppose you imbedded some free charge in a piece of glass. About how long would it take for the charge to flow to the surface?

The equation that governs how charge dissipates through a conductor is given by Equation 9.120 as

$$\rho_f = e^{-(\sigma/\varepsilon)t} \rho_f(0)$$

thus any initial free charge densiy $\rho_f(0)$ dissipates in a characteristics time

$$\tau \equiv \frac{\varepsilon}{\sigma}$$

where

$$\frac{1}{\sigma_{glass}} = \rho_{glass} \approx (10^{10} - 10^{14})$$

where ρ_{glass} is the resitivity and σ_{glass} is the conductivity and the permitivity is

$$\varepsilon_{glass} \approx 4.9 - 7.5\varepsilon_0$$

we find that the characteristic time is given by

$$\begin{aligned} \tau &\approx 0.433\,\text{sec} \quad \frac{1}{\sigma_{glass}} \approx 10^{10} \quad \varepsilon_{glass} \approx 4.9\varepsilon_0 \\ \tau &\approx 6198\,\text{sec} \quad \frac{1}{\sigma_{glass}} \approx 10^{14} \quad \varepsilon_{glass} \approx 7.0\varepsilon_0 \end{aligned}$$

b) Silver is an excellent conductor but it's expensive. suppose you were designing a microwave experiment to operate at a frequency of 10^{10} Hz. How thick would you make the silver coatings?

We know that the distance it takes to reduce the amplitude of a wave by a factor of $1/e$ (about a third) is called the **skin depth** which is defined as

$$d \equiv \frac{1}{\kappa} \tag{9.8}$$

where κ is defined as

$$\kappa \equiv \omega \sqrt{\frac{\varepsilon\mu}{2}} \left[\sqrt{1 + \left(\frac{\sigma}{\varepsilon\omega}\right)^2} - 1 \right]^{1/2} \tag{9.9}$$

thus if the thickness of our silver sheet is given by 8 we will have a coating with a thicknes that will stop the microwaves by at least 1/3. Given that

$$\mu_{silver} \approx \mu_0 \quad \frac{1}{\sigma_{silver}} = 1.59 \times 10^{-8} \quad \varepsilon_{silver} = \varepsilon_0$$

since we also know that $\omega = 2\pi f$ where f is the frequency we find

$$\kappa = 1.57 \times 10^6 \text{ m}^{-1}$$

thus we should make this

$$d \approx 6.36 \times 10^{-7}\text{m}$$

we should probably make it a little thicker than this to stop all EM waves, i.e

$$\boxed{\text{width} \approx 2d \approx 1.27 \times 10^{-6}\text{m}}$$

c) Find the wavelength and propgation speed in copper for radio waves at 1 Mhz. Compare the corresponding values in air (or vacuum).

We know that the wavelength and speed of propgation is given by

$$\lambda = \frac{2\pi}{k} \quad v = \frac{\omega}{k}$$

where k is defined as

$$k \equiv \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}+1\right]^{1/2}$$

for copper

$$\frac{1}{\sigma_{copper}} = \rho_{copper} \approx 1.68 \times 10^{-8} \quad \omega = 2\pi f = 2\pi \times 10^6\text{s}^{-1}$$

thus

$$\sigma_{copper} \approx 5.9 \times 10^7$$

we also know that $\varepsilon_{copper} \approx (2-20)\varepsilon_0$

$$\sigma \gg \varepsilon\omega$$

we can know that

$$\left(\frac{\sigma}{\varepsilon\omega}\right)^2 \gg 1 \;\Rightarrow \sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2} \gg 1$$

and thus we can write

$$k \approx \omega\sqrt{\frac{\varepsilon\mu}{2}}\sqrt{\frac{\sigma}{\varepsilon\omega}} \approx \sqrt{\frac{\sigma\omega\mu}{2}}$$

where $\mu \approx \mu_0$ we find

$$k \approx \sqrt{\frac{\sigma\omega\mu_0}{2}} \approx 1.5 \times 10^4\text{m}^{-1}$$

thus the wavelength is

$$\boxed{\lambda = \frac{2\pi}{k} \approx 4.11 \times 10^{-4}\text{m}}$$

and the velocity is given as

$$\boxed{v = \frac{\omega}{k} \approx 419 \text{ m/s}^{-1}}$$

Problem # 6

A plane wave travels in the $\hat{\mathbf{z}}$ direction in a conductor with real conductivity.

a) Find the instantaneous and time-averaged power loss per unit volume due to resistive heating for any z.

The electric field and magnetic fields in this case are given by Equation 9.138 in Griffiths as

$$\begin{aligned} \mathbf{E}(z,t) &= E_0 e^{-\kappa z}\cos(kz-\omega t+\delta_E)\hat{\mathbf{x}} \\ \mathbf{B}(z,t) &= B_0 e^{-\kappa z}\cos(kz-\omega t+\delta_E+\phi)\hat{\mathbf{y}} \end{aligned} \tag{9.10}$$

The power loss due to resistive heating is given as

$$dP = d(VI)$$

we know that

$$dV = Edx$$

and

$$dI = J\cdot d\mathbf{a} = Jdydz$$

thus

$$dP = JEdxdydz$$

where

$$J = \sigma E$$

thus the power loss per unit volume is given by

$$\boxed{\frac{P}{V} = \sigma E^2 = \sigma E_0^2 e^{-2\kappa z}\cos^2(kz-\omega t+\delta_e)}$$

and the time average power loss is given as

$$\boxed{\langle\frac{P}{V}\rangle = \frac{\sigma}{2}e^{-2\kappa z}E^2}$$

because we know that

$$\langle\cos^2(kz-\omega t+\delta_E)\rangle = \frac{1}{2}$$

b) Find the total power loss per unit area between $z = 0$and $z \to \infty$.

To find the total power loss per unit area we must integrate the power per volume

$$\frac{P}{A} = \int_0^\infty \langle\frac{P}{V}\rangle dz = \frac{1}{2}\sigma E_0^2\int_0^\infty e^{-2\kappa z}dz = \boxed{\frac{\sigma}{4\kappa}E_0^2 e^{-2\kappa z}}$$

c) Find the time-averaged Poynting vector at any z.

The Poynting vector is defined as

$$\mathbf{S} = \frac{1}{\mu}(\mathbf{E}\times\mathbf{B})$$

using Equation 10 and Equation 11 we find

$$\mathbf{S} = \frac{B_0E_0}{\mu}e^{-2\kappa z}[\cos(kz-\omega t+\delta_E)\cos(kz-\omega t+\delta_E+\phi)]\hat{\mathbf{z}}$$

if we let $x = kz - \omega t + \delta_E$ and $\phi = \pi/4$ we find

$$\mathbf{S} = \frac{B_0 E_0}{\mu} e^{-2\kappa z}[\cos(x)\cos(x+\phi)]\hat{\mathbf{z}} = \frac{B_0 E_0}{\mu} e^{-2\kappa z}[\cos^2 x \cos\phi - \cos x \sin x \sin\phi]\hat{\mathbf{z}}$$

The intensity is given as

$$\mathbf{I} = \langle \mathbf{S} \rangle = \frac{B_0 E_0}{2\mu} e^{-2\kappa z} \cos\phi \hat{\mathbf{z}} = \frac{E_0^2}{2\omega\mu} e^{-2\kappa z} K \cos\phi \hat{\mathbf{z}}$$

since we know that

$$\tan\phi = \frac{\kappa}{k} \qquad \cos\phi = \frac{k}{\sqrt{k^2+\kappa^2}}$$

and since we know that

$$K = \sqrt{k^2+\kappa^2}$$

we find

$$K\cos\phi = k$$

thus the intensity is given as

$$\mathbf{I} = \langle \mathbf{S} \rangle = \frac{k}{\omega\mu}\frac{E_0^2}{2} e^{-2\kappa z}\hat{\mathbf{z}}$$

d) Compare the value of your result for parts (b) and (c). Are the results in accord with one another? Explain.

at $z = 0$ the magnitude of these two functions should be the same. For the Poynting Vector we find

$$\mathbf{S}(z=0) = \frac{kE_0^2}{2\mu\omega}$$

and for the energy per unit area per unti volume we find

$$\frac{P(z=0)}{A} = \frac{\sigma}{4\kappa} E_0^2$$

we need to show

$$\frac{k}{\mu\omega} = \frac{\sigma}{2\kappa}$$

this can be done using

$$k\kappa = \omega^2 \frac{\varepsilon\mu}{2}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}+1\right]^{1/2}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}-1\right]^{1/2} = \omega^2\frac{\varepsilon\mu}{2}\frac{\sigma}{\varepsilon\omega}$$

thus we find

$$\frac{k}{\mu\omega} = \frac{\sigma}{2\kappa}$$

which is what we were looking for all along thus these two quantities are equal at $z = 0$

Problem # 7 A parallel-plate capacitor at high frequencies

When one designs an electronics circuit with conventional devices like resistors, capacitors and inductors, one is wise to keep in mind that "everything changes at high frequencies". To illustrate this, let us consider a capacitor composed of two closely spaced and parallel circular plates, with radius a and separation h ($h \ll a$).

a) Consider first the "quasi-static" situation, by which we mean the low-frequency response of the capacitor. If the capacitor is driven with a single-frequency sinusoidal voltage $V = V_0 e^{i\omega t}$, what are the electric and magnetic fields, $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$, produced inside the capacitor? You can write these as complex quantities, but keep in mind that the physical values of fields, etc., are given by the real parts of the complex functions.

We know that

$$\mathbf{E}(\mathbf{r},t) = \frac{V}{h}\hat{\mathbf{z}} = V_0\frac{e^{i\omega t}}{h}\hat{\mathbf{z}} = \frac{V_0}{h}(\cos\omega t + i\sin\omega t)\hat{\mathbf{z}}$$

thus the real part is given by

$$\boxed{\mathbf{E}(\mathbf{r},t) = \frac{V_0}{h}\cos\omega t\hat{\mathbf{z}}}$$

we also know from Maxwell's equation

$$\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{B}$$

and using Stoke's theorem

$$\frac{1}{c^2}\int \mathbf{E}\cdot d\mathbf{a} = \oint \mathbf{B}\cdot d\mathbf{l}$$

we know

$$d\mathbf{a} = \pi r^2 \quad d\mathbf{l} = 2\pi r\hat{\phi}$$

thus we have

$$\frac{\pi r^2}{c^2}\frac{d\mathbf{E}}{dt} = \mathbf{B}2\pi r$$

where

$$\frac{d\mathbf{E}}{dt} = -\frac{V_0}{h}\omega\sin\omega t$$

thus we find the magntic field to be given as

$$\boxed{\mathbf{B}(\mathbf{r},t) = -\omega V_0\frac{r}{2hc^2}\sin\omega t\hat{\phi}}$$

b) From the formula relating the magnetic energy and self-inductance of an inductor and from your answer above, what is the self-inductance for this capacitor? In determining this energy, you will have to consider a specific volume over which to integrate the energy density - take this to be the inner volume of the capacitor.

We know that the energy is given by

$$W = \frac{1}{2}LI^2$$

Equation 7.34 gives the energy of the magnetic field as

$$W = \frac{1}{2\mu_0}\int_{all\ space} B^2 d\tau$$

thus

$$\frac{1}{2\mu_0}\int_{all\ space} B^2 d\tau = \frac{1}{2}LI^2$$

the left hand side becomes

$$\frac{1}{2\mu_0}\int\left(\omega V_0 \frac{r}{2hc^2}\sin\omega t\right)^2 d\tau \quad d\tau = 2\pi rhdr$$

which gives

$$\frac{1}{2\mu_0}\int\left(\omega V_0 \frac{r}{2hc^2}\sin\omega t\right)^2 2\pi hrdr = \frac{h\pi}{\mu_0}\left(\frac{\omega V_0}{2hc^2}\right)^2 \sin^2\omega t\int_0^a r^3 dr$$

thus

$$W = \frac{\pi a^4}{4\mu_0 h}\left(\frac{\omega V_0}{2c^2}\right)^2 \sin^2\omega t$$

to find the inductance

$$\frac{\pi a^4}{4\mu_0 h}\left(\frac{\omega V_0}{2c^2}\right)^2 \sin^2\omega t = \frac{1}{2}LI^2 \tag{9.11}$$

if we let

$$I = C\frac{dV}{dt} \quad C = \frac{\pi a^2\varepsilon_0}{h} \quad \frac{dV}{dt} = -\omega\frac{V_0}{h}\sin\omega t$$

thus

$$I^2 = \left(\frac{\pi a^2\varepsilon_0}{h}\frac{V_0}{h}\sin\omega t\right)^2$$

substituting this into Equation 15 gives a inductance of

$$\boxed{L = \frac{\mu_0 h}{8\pi}}$$

c) From the standard capacitance C and the inductance L that you just calculated, determine a resonant frequency ω_0 for the capacitor as you would for a standard $L-C$ circuit. The frequency ω_0 is a good benchmark for what qualifies as “low” or quasi-static vs. “high” frequency operation of the capacitor.

We know that for n L-C circuit the angular frequency is defined as

$$\omega = \frac{1}{\sqrt{LC}}$$

we know that

$$LC = \frac{a^2}{8c^2}$$

thus the resonant frequency is given by

$$\boxed{\omega_0 = \frac{c}{a}\sqrt{8}}$$

d) In (a), you will have determined the magnetic field generated by the time-varying electric field from Faraday’s law. But, of course, the time-varying magnetic field also generates an electric field according to the Ampere-Maxwell law. Taking the magnetic field discussed in parts (a) and (b), calculate this correction to the electric field of a capacitor.

The Ampere-Maxwell law gives

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$$

using Stoke's theorem we find

$$\oint \mathbf{E}\cdot d\mathbf{l} = -\frac{d}{dt}\int \mathbf{B}\cdot d\mathbf{a}$$

where

$$d\mathbf{a} = hdr\hat{\phi}$$

and using the magnetic field from part a) we find

$$E(0)h - E(r)h = \frac{d}{dt}\int_0^r \omega V_0 \frac{r}{2hc^2}\sin\omega t(hdr)$$

where

$$\frac{d}{dt}\int_0^r \omega V_0 \frac{r}{2hc^2}\sin\omega t(hdr) = \frac{\omega^2 V_0}{4c^2}r^2\cos\omega t$$

defining

$$E(0) = 0$$

we find

$$\boxed{E(r) = -\frac{\omega^2 V_0}{4hc^2}r^2\cos\omega t\hat{\mathbf{z}}}$$

this is the correction to the electric field due to the changing magnetic field. Thus the total electric field is

$$E_{tot} = \left(\frac{V_0}{h} - \frac{\omega^2 V_0}{4hc^2}r^2\right)\cos\omega t\hat{\mathbf{z}}$$

e) You should find that, at this level of approximation, the electric field inside the capacitor is parabolic as a function of radius r. At what frequency ω does the electric field at the outer edge of the capacitor a go to zero? How does it compare with the frequency ω_0 determined above?

If we set the electric field equal to zero we find

$$\frac{V_0}{h} - \frac{\omega^2 V_0}{4hc^2}r^2 = 0$$

we find that the frequency that makes this go to zero is

$$\boxed{\omega = \frac{2c}{a}}$$

which is smaller by $\sqrt{2}$.

Beyond the approximation discussed above, a complete solution for the electric and magnetic fields inside a capacitor can be found that is consistent with MaxwellÕs equations. As often in problems with cylindrical symmetry, the solutions for the spatial variations of E and B are Bessel functions. These functions also serve to characterize the resonant modes inside a cylindrical metal can.